High order schemes for gradient flows and vaccination mean field games

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Outline

Gradient flows: theory

• $F : \mathbb{R}^d \to \mathbb{R}$ = a smooth convex function, $\bar{x} \in \mathbb{R}^d$; gradient flow from $\bar{x}$ = a curve $(x_t)_{t \geq 0}$: $x_t' = -\nabla F(x_t)$ for $t > 0$, $x_0 = \bar{x}$.
• Polish metric space $(\mathcal{X}, d)$, functional $F : (\mathcal{X}, d) \to \mathbb{R} \cup \{+\infty\}$: non-trivial defintion, huge litterature (cf. books by Ambrosio et al. , Villani, Santambroggio).

• Euclidian space (under some regularity assumptions):

$$\frac{d}{dt}F(x_t) = \langle \nabla F(x_t), x_t' \rangle \geq -|x_t'| \cdot |\nabla F|(x_t) \geq -\frac{1}{2}|x_t'|^2 - \frac{1}{2}|\nabla F|^2(x_t),$$

$$\text{or equivalently } \frac{d}{dt}F(x_t) + \frac{1}{2}|x_t'|^2 + \frac{1}{2}|\nabla F|^2(x_t) \geq 0,$$

with equality only if $x_t' = -\nabla F(x_t)$.
• Conclusion: $\frac{d}{dt}F(x_t) + \frac{1}{2}|x_t'|^2 + \frac{1}{2}|\nabla F|^2(x_t) \leq 0$ a.e. is equivalent with $x_t' = -\nabla F(x_t)$.

Gradient flows: theory

- Euclidian space formulation: $\frac{d}{dt}F(x_t) + \frac{1}{2}\left|x_t'\right|^2 + \frac{1}{2}\left|\nabla F\right|^2(x_t) \leq 0$ a.e.
- the (local metric) slope of F at x:

$$\left|\nabla F\right|(x) = \limsup_{z\to x} \frac{[F(x)-F(z)]_+}{d(x,z)} = \max\left\{\limsup_{z\to x} \frac{F(x)-F(z)}{d(x,z)}, 0\right\}.$$

- the metric derivative of x at t: $|x_t'| = \lim_{h\to 0} \frac{d(x_{t+h},x_t)}{|h|}$, exists a.e. as soon as $t \mapsto x_t$ is absolutely continuous. Moreover $|x'| \in L^1(0,1)$.
- EDI ∇-flow (pointwise): $\frac{d}{dt}F(x_t) + \frac{1}{2}\left|x_t'\right|^2 + \frac{1}{2}\left|\nabla F\right|^2(x_t) \leq 0$ a.e.
- EDI ∇-flow from $\bar{x}$: an absolutely continuous curve such that:

$$\forall s \geq 0,\ F(x_s) + \frac{1}{2}\int_0^s |x_r'|\,\mathrm{d}r + \frac{1}{2}\int_0^s |\nabla F|^2(x_r)\,\mathrm{d}r \leq F(\bar{x}),$$

$$\text{a.e. } t > 0,\ \forall s \geq t,\ F(x_s) + \frac{1}{2}\int_t^s |x_r'|\mathrm{d}r + \frac{1}{2}\int_t^s |\nabla F|^2(x_r)\mathrm{d}r \leq F(x_t).$$

- EVI form for λ-convex (i.e., when smooth $F'' \geq \lambda Id$ ...) functionals: $F(x_t) + \frac{1}{2}\frac{d}{dt}d^2(x_t,y) + \frac{\lambda}{2}d^2(x_t,y) \leq F(y), \forall y,\ a.e.\ t \geq 0.$

Gradient flows examples: heat flow (Fokker-Planck)

$\mathcal{X} = \mathcal{P}_2(\mathbb{R})$ (the set of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite second-order moment, endowed with the Wasserstein distance $\mathcal{W}_2$)
Consider for $\sigma \in \mathbb{R}$ $F : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$:
$F(\nu) = \int_{\mathbb{R}} V(x)\rho(x) + \frac{\sigma^2}{2} \int_{\mathbb{R}} \rho(x) \log(\rho(x))dx$, if $\nu \ll dx$, $\nu = \rho(x)dx$
$F(\nu) = +\infty$, if $\nu \not\ll dx$.
For smooth V, the gradient flow $t \mapsto \nu(t) \in \mathcal{P}_2(\mathbb{R})$ of F satisfies $\nu(t) = \rho(t, \cdot)dx$ and:

$$\frac{\partial \rho}{\partial t}(t,x) = \frac{\partial}{\partial x}[V'(x)\rho(t,x)] + \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2}(t,x), \qquad (1)$$

i.e., Fokker-Planck of the SDE: $dX(t) = -V'(X(t))dt + \sigma dW(t)$.

Remark: also a L^2 flow (term $\int |\nabla \rho|^2$)...

Gradient flows examples: heat flow (Fokker-Planck)

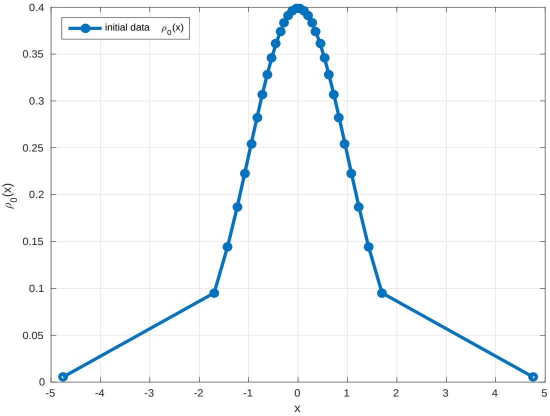


Figure: Initial data for the heat flow (FP) model and its evolution (VIDEO).

Gradient flows examples : a 1D Patlak-Keller-Segel model

• the (modified) Patlak–Keller–Segel system (Perthame-Calvez-Sharifi Tabar 2007, Blanchet-Calvez-Carrillo 2008), is a PDE model for diffusion-aggregation competition in biological applications (chemotaxis).

• Free energy functional:

$$\mathcal{G}[\rho] = \int \rho(t,x)\log(\rho(t,x))\,\mathrm{d}x + \frac{\chi}{\pi}\int\int \rho(t,x)\rho(t,y)\log|x-y|\,\mathrm{d}x\mathrm{d}y$$

• the resulting Patlak-Keller-Segel equation:

$$\frac{\partial\rho}{\partial t} = \Delta\rho - \nabla(\chi\,\rho\nabla c),\ t > O,\ x \in \Omega \subset \mathbb{R}^d \qquad (2)$$

$$c = -\frac{1}{d\pi}\log|z| \star \rho \qquad (3)$$

ρ = cell density, c = concentration of chemo-attractant, χ = sensitivity of the cells to the chemo-attractant.

Gradient flows examples: 1D Patlak-Keller-Segel model

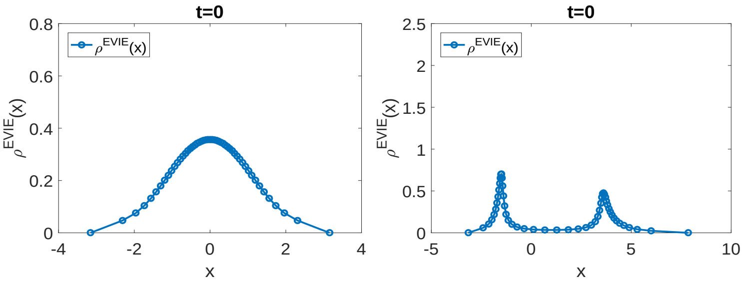


Figure: Initial data for the PKS model: $\chi = \pi$ (left), $\chi = 1.9\pi$ (right) and its evolution (VIDEO $T = 2$). Implementation : G. Legendre; ∇-flow JKO PKS code : courtesy A. Blanchet.

Gradient flows: the JKO scheme

• Jordan, Kinderlehrer and Otto '98, (JKO) numerical scheme: time step $= \tau > 0$, $x_0^\tau = \bar{x} \in \mathcal{X}$, by recurrence $x_{n+1}^\tau =$ a minimizer of the functional

$$x \mapsto P_F^{JKO}(x; x_n^\tau, \tau) := \frac{1}{2\tau} d^2(x_n^\tau, x) + F(x). \tag{4}$$

• If $\mathcal{X}=$ Hilbert, $F=$ smooth, JKO $=$ implicit Euler (IE) scheme, i.e., $\frac{x_{n+1}^\tau - x_n^\tau}{\tau} = -\nabla F(x_{n+1}^\tau)$.

• JKO scheme was initially used theoretically to prove the existence of a gradient flow

• JKO scheme only (!!) was then used numerically to compute the gradient flow (J.K.O '99, Blanchet et al. 2009, Benamou et al 2016, ...). What about other numerical schemes ?

• JKO $=$ first-order. Dynamics is regular with respect to time ! What about higher (second) order ?

High order schemes

• Idea 1: by Runge-Kutta : ODE $x'(t) = f(x(t))$ $(f = -\nabla F)$
Crank-Nicholson $x_{k+1}^{\tau} = x_k^{\tau} + \tau \frac{f_{k+1}+f_k}{2}$, 2nd order.
For ∇-flows in a metric space: no gradient 'f', no vector calculus.

• Idea 2 (from symplectic integrators): increase the order by composition
⋆ here: take F quadratic, equation $x'(t) = Ax(t)$ (linear);
⋆ Implicit Euler: $\frac{x_{k+1}^{\tau}-x_k^{\tau}}{\tau} = Ax_{k+1}^{\tau}$ thus $x_{k+1}^{\tau} = (I - \tau A)^{-1} x_k^{\tau}$;
⋆ composition of IE steps $\alpha_1 h$, ... $\alpha_n h$: multiplication by
$(I - \alpha_n \tau A)^{-1} ... (I - \alpha_1 \tau A)^{-1}$;
⋆ condition to be the same as $exp(A\tau)$:
first order $\sum_{\ell=1}^{n} \alpha_\ell = 1$;
second order $\sum_{\ell=1}^{n} \alpha_\ell^2 + \sum_{1\le \ell < m \le n} \alpha_\ell \alpha_m = 1/2$.
Second condition implies $\left(\sum_{\ell=1}^{n} \alpha_\ell\right)^2 - \sum_{1\le \ell < m \le n} \alpha_\ell \alpha_m = 1/2$ thus
$\sum_{1\le \ell < m \le n} \alpha_\ell \alpha_m = 1/2$, $\sum_{\ell=1}^{n} \alpha_\ell^2 = 0$.

CANNOT obtain second order from composition of I.E. schemes.

Second order schemes for gradient flows: the VIM scheme

• Recall for ODE $x'(t) = f(x(t))$: consistency error ($t_k = k\tau$).
consistency error for Implicit Euler $\frac{x(t_{k+1}) - x(t_k)}{\tau} - f(x(t_{k+1})) =$
$x'(t_{k+1/2}) + O(\tau^2) - f(x(t_{k+1})) = \underbrace{f(x(t_{k+1/2})) - f(x(t_{k+1}))}_{O(\tau)} + O(\tau^2)$.

• (modified) Midpoint method: $x^\tau_{k+1} = x^\tau_{k-1} + 2\tau f_k$, 2nd order.

• idea: use the variational formulation : Variational Implicit Midpoint (VIM) scheme (G. Legendre, G.T., '16):
$x^\tau_{k+1} \in argmin_{x\in\mathcal{A}} \frac{d(x^\tau_k, y)^2}{2\tau} + 2F(\frac{x^\tau_k + y}{2})$

"$\frac{x+y}{2}$" = the midpoint of the geodesic from x to y.
• Hilbert space critical point equation: $\frac{x^\tau_{n+1} - x^\tau_n}{\tau} + \nabla F(\frac{x^\tau_{n+1} + x^\tau_n}{2}) = 0$.
Consistency error = $O(\tau^2)$.

Second order schemes for gradient flows: the EVIE scheme

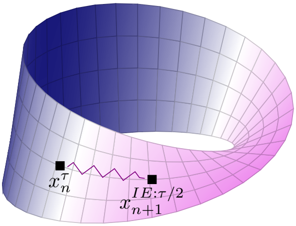
• Re-writing of the VIM scheme: $x_{k+1}^\tau \in argmin_{x\in\mathcal{A}} \frac{d(x_k^\tau,y)^2}{2\tau} + 2F(\frac{x_k^\tau+y}{2})$

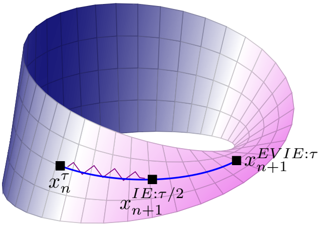
• Notation $z = \frac{x_k^\tau+y}{2}$, then y is the 2-geodesic-extrapolate of x_k^τ with respect to z, "$y = 2z - x_k^\tau$"; $d(x_k^\tau, y) = 2d(x_k^\tau, z)$.

• $\min_{z\in\mathcal{A}...} \frac{d(x_k^\tau,z)^2}{2(\tau/2)} + F(z)$: BUT this is I.E. of step $\tau/2$!

• Extrapolated Variational Implicit Euler (EVIE) scheme : do a $\tau/2$ IE (= JKO) step and then extrapolate on the geodesic.

EASY to implement in an existing JKO / IE code ! OK in Hilbert spaces ...





Numerical results for VIM and EVIE schemes: heat flow

Numerical results for $F(\nu) = \int_{\mathbb{R}} V(x)\rho(x) + \frac{\sigma^2}{2}\int_{\mathbb{R}} \rho(x)\log(\rho(x))dx$, $V(x) = \theta\frac{(x-\mu)^2}{2}$, $T = 1$, $\sigma = 1$, $\theta = \frac{1}{2}$, $\mu = 5$, $M = 32$ spatial discretization points.

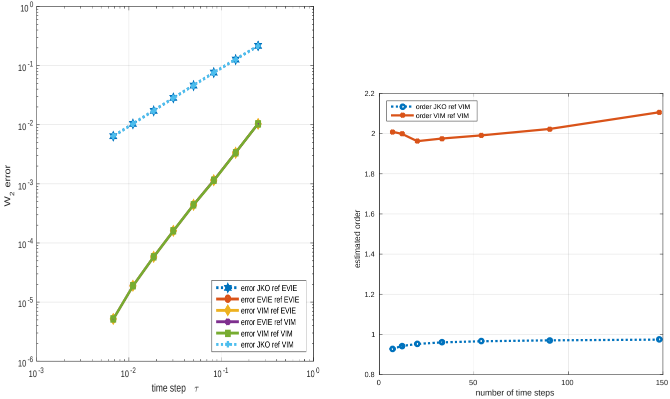


Figure: # of time steps: 4, 7, 12, 20, 33, 54, 90 and 148 (reference 244). **Left:** error for JKO (dotted line) and VIM / EVIE (solid lines) schemes. **Right:** order of convergence: JKO (dotted line), VIM / EVIE (solid lines). 4 steps VIM/EVIE = 90 steps JKO/IE.

Numerical results for the EVIE scheme: PKS

$\mathcal{G}[\rho] = \int \rho(t,x)\log(\rho(t,x))\,\mathrm{d}x + \frac{\chi}{\pi}\int\int \rho(t,x)\rho(t,y)\log|x-y|\,\mathrm{d}x\mathrm{d}y$

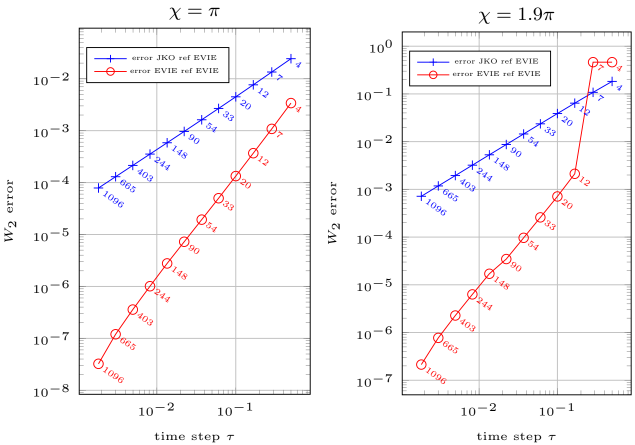


Figure: Error of the JKO and EVIE schemes for PKS model; $T = 2$, time steps: reference sol EVIE(1808). **Left:** $\chi = \pi$, order JKO $= 1.02$, order EVIE $= 2.01$; **Right:** $\chi = 1.9\pi$, order JKO $= 0.99$, order EVIE $= 2.00$ (corrected by excluding first two points).

Theoretical results for the VIM scheme

mid-slope : $\left|\nabla^M F\right|(x,y) = \limsup_{z\to y} \frac{\left(F\left(\frac{x+y}{2}\right) - F\left(\frac{x+z}{2}\right)\right)^+}{d\left(\frac{x+y}{2}, \frac{x+z}{2}\right)}$.

Hypothesis (non standard):

- (geometric) $\forall x \in \mathcal{X}$ the set $\bigcup_{y\in\mathcal{X}} \frac{x+y}{2}$ is closed;
- (geometric) $\forall (x,y) \in \mathcal{X}^2$, the set $\frac{x+y}{2}$ is a singleton.
- (adaptation for $\left|\nabla^M F\right|$ instead of $|\nabla F|$) $\forall x \in D(F)$, $D(F) \supset (x_n)_{n\in\mathbb{N}} \to x$ and $D(F) \supset (y_n)_{n\in\mathbb{N}} \to x$ imply $\left|\nabla^M F\right|(x,x) \le \liminf_{n\to\infty} \left|\nabla^M F\right|(x_n, y_n)$;
- (regularity for F) if any two of the elements x, y, $\frac{x+y}{2}$ belong to $D(F)$, then the third also does and:

$$\left|\frac{F(x) + F(y) - 2F\left(\frac{x+y}{2}\right)}{d^2(x,y)}\right| \le H, \tag{5}$$

where H is a constant independent of x and y.

Sufficient condition: F and $-F$ are λ-convex.

Theoretical results for the VIM scheme

Hypothesis (standard):

• F is lower semicontinuous, bounded from below, and such that: $\forall r > 0, \forall c \in \mathbb{R}, \forall x \in \mathcal{X}$ the set $\{y \in \mathcal{X} \mid F(y) \leq c,\ d(x,y) \leq r\}$ is compact,
• F has the following continuity property
if $x_n \to x$, and $\sup\{|\nabla F|(x_n), E(x_n)\} < \infty$ then $F(x_n) \to F(x)$;

Theorem (G. Legendre, G.T. 2016)

Let $T > 0$ be fixed and $(\mathcal{X}, d)$ be a Polish metric space. Under above hypotheses for some $\bar{\tau} > 0$, the set of curves $\{(x_t^\tau)_{t \in [0,T]}; 0 \leq \tau \leq \bar{\tau}\}$ is relatively compact (with respect to the local uniform convergence) and any limit curve is a gradient flow in the EDI formulation.

• this is consistency
• what about the (second) order of convergence ?

Outline

Disclaimer:

What follows is a THEORETICAL epidemiological investigation. It is not meant to be used directly for health-related decisions; if in need to take such a decision please seek professional medical advice.

Vaccine scares: MFG models

Influenza A (H1N1) (flu) (2009-10)

- At 15/06/2010 flu (H1N1): 18.156 deaths in 213 countries (WHO)
- France: 1334 severe forms (out of $7.7M$-$14.7M$ people infected)

Countries	Official target coverage	Effective rate of vaccination
Germany	100 %	10%
Belgium	100 %	6 %
Spain	40 %	$< 4\%$
France	70 - 75 %	8.5 %
Italy	40 %	1.4 %

Previous vaccine scares (some have been disproved since):

- France: hepatitis B vaccines cause multiple sclerosis
- US: mercury additives are responsible for the rise in autism
- UK: the whooping cough (1970s), the measles-mumps-rubella (MMR) (1990s).

Further motivation: end of compulsory vaccination

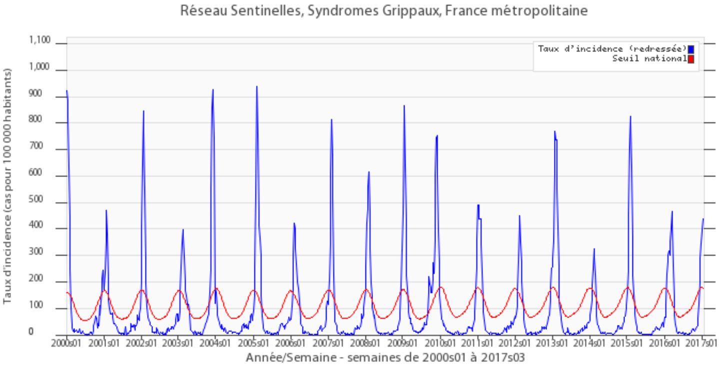
• context in France: discussions on the end of general compulsory vaccination

• How is vaccination coverage evolving ? Example: nobody vaccinates then an additional individual may vaccinate; if all vaccinate an additional individual will not vaccinate. Will the vaccination coverage become unstable or chaotic ?

• question: what are the determinants of individual vaccination

• hint: individual decisions sum up to give a global response; need a model.

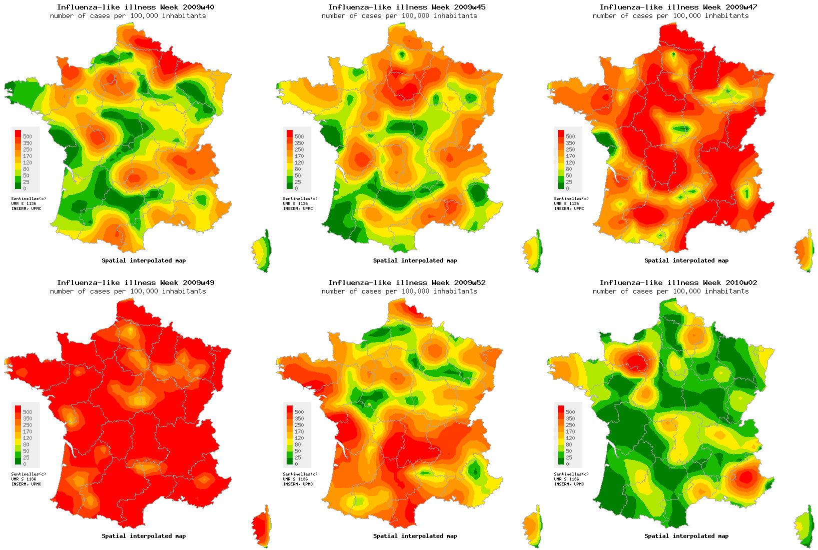
Influenza incidence historical data in France



Sourse: 'Sentinelles' network, INSERM/UPMC, http://www.sentiweb.fr

Influenza A (H1N1) (flu) (2009-10) : see next.

Influenza A (H1N1) (flu) (2009-10), France



The SIR-V model

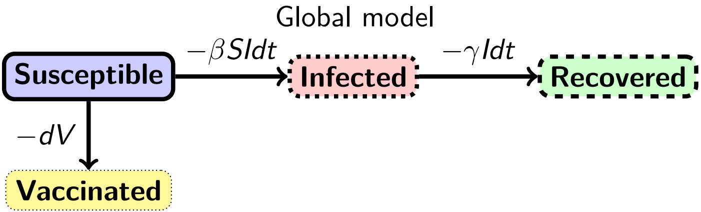
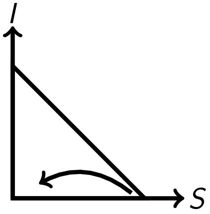


Figure: Graphical illustration of the SIR-V model. In this model all individuals are identical. β : probability of contamination, γ : recovery rates, $dV(t)$: measure of vaccination, can be sum of Diracs (!!)

$$\begin{cases} dS = -\beta SIdt - dV(t) \\ \frac{dI(t)}{dt} = \beta SI - \gamma I \\ \frac{dR(t)}{dt} = \gamma I(t) \end{cases}$$



Vaccination cost functional

Cost for an infected person : r_I.
Cost of the vaccine (including side-effects) : r_V.

Global cost for the society (from the initial state $X_0 = (S(0), I(0))^T$) :

$$J(X_0, V) = r_I \int_0^\infty \beta SIdt + r_V \int_0^\infty dV(t) \tag{6}$$

• General tools: Abakuks, Andris, 1974 "*Optimal immunisation policies for epidemics*", ...
• viscosity solution (HJB version): L. Laguzet & GT, Math. Biosci., 263:180–197, 2015

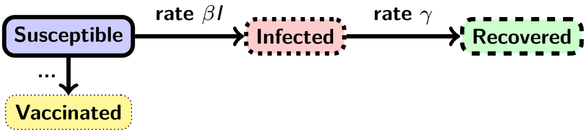
This gives the "official" targets, BUT is not what individuals do !

Taking into account the individual decisions: previous literature

Bauch & Earn (PNAS 2004) : a SIR-V model with births and deaths, constant vaccination rate
Results : no eradication possible, ...
Francis (2004) : uses a SIR-V model to find an equilibrium.
Galvani, Reluga & Chapman (PNAS 2006) consider a double SIR-V periodic model of flu with two age groups (break at 65yrs). Vaccination is separated from dynamics, once at the beginning of each season.
Results : show that actual vaccine coverage is consistent with individual optimum; explain impact of age-targetted campaigns (children).
Bauch (2005) : time dependent vaccination rate, the corresponding dynamics is a phenomenological proposal
B. Buonomo : vaccination as a feedback
F. Fu : taking into account the topology (networks of acquaintances)

Individual dynamics

Global dynamics : continuous time deterministic ODE; is the master equation of the individual dynamics.



Individual dynamics: continuous time Markov jumps between 'Susceptible', 'Infected', 'Recovered' and 'Vaccinated' classes.

Individual cost functional (discount = 0)

$$\begin{cases} dS = -\beta SIdt - dV(t) \\ dI = (\beta SI - \gamma I)dt. \end{cases}$$

vaccination at the society level with $dV(t)$; it can have the form $dV(t) = u_G(t)dt$.

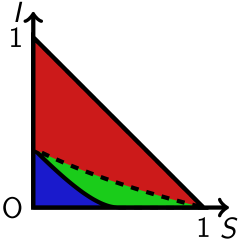
- Society dynamics induces a cumulative probability of infection on $[0,t]$: $\varphi_I(t)$ solution of $d\varphi_I(t) = \beta I(t)(1-\varphi_I(t))dt$, $\varphi_I(0) = 0$.
- Individual decision: φ_V (with constraints cf. above).
- Individual cost functional : depends on the individual policy $d\varphi_V$, a probability law

$$J_{indi}(\varphi_V; V) = r_I \varphi^I(\infty) + \int_{0^-}^{\infty} \Big[r_V - r_I \varphi^I(\infty) + (r_I - r_V)\varphi^I(t) \Big] d\varphi_V(t).$$

- Individual - global equilibrium condition (global strategy arise from individual decisions): $dV(t) = \frac{S(t)}{1-\varphi_V(t)} d\varphi_V(t)$ (when this operation makes sense) (Mean Field Games, cf. Lasry - Lions, Caines - Huang - Malhamé).
- related works: C. Gueant (graphs), O. Cardaliaguet (HJB), D. Gomes : hyp. of superlinear costs (here linear); pure strategy set is not compact.

Theoretical results for discount=0

Analytic description of the Nash-MFG equilibrium (G.T., L. Laguzet '15)

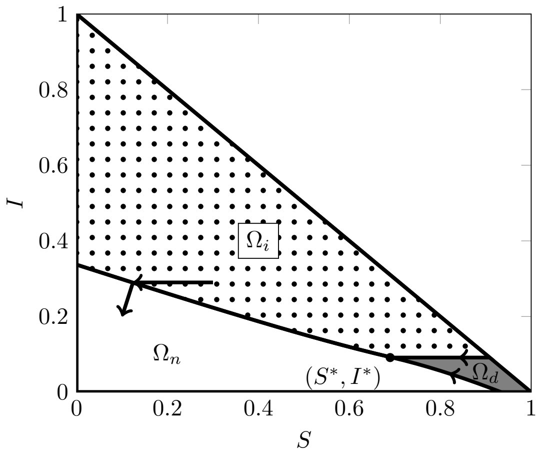


Red region: vaccination in the societal model and in the individual model, green region : vaccination for the societal model but not for the individual, blue region: no vaccination in both models.

- ⋆ Mean cost for an individual in the stable individual-global equilibrium is larger than the cost for the, non equilibrium, societal (non-individual) optimum state
- ⋆ this is because in the green region, it is optimal for individuals to let other vaccinate.
- ⋆ Conclusion: the stable strategy will be obtained even if it is more costly for everyone; the "cost of anarchy" in the model is strictly > 0.

Equilibrium for discount > 0, no constraints on vaccination speed

Theoretical results (GT, LL 2015)



White region: no vaccination, Π_∞

gray region : delayed vaccination (new behavior, Π_{t^*})

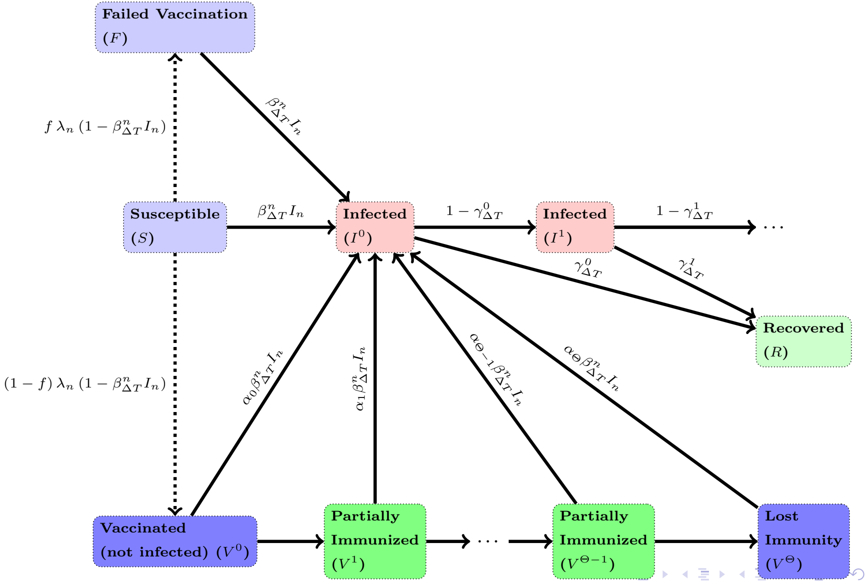
dotted region: immediate vaccination, Π_0

Comparing Π_0 and Π_∞ (Francis) is not enough, more complicated dynamics than $\mathcal{D} = 0$.

More complicated regions for finite vaccination speed(L. Laguzet, G. Yahiaoui, G. T. 2016), uniqueness not proved.

Imperfect vaccination: efficacy, delays (with F. Salvarani)

The individual model in DISCRETE TIME



Imperfect vaccination

• Continuous time-version:

$S'(t) = -\beta S(t)I(t) - U'(t)$

$I'(t) = \beta \left[S(t) + F(t) + \int_0^{+\infty} A(\theta)V(t,\theta)d\theta \right] I(t) - \gamma I(t)$

$\partial_t V(t,\theta) + \partial_\theta V(t,\theta) = -\beta A(\theta)V(t,\theta)I(t)$

$F'(t) = fU'(t) - \beta F(t)I(t).$

• Initial and boundary conditions

$S(0) = S_{0^-}, I(0) = I_{0^-}, \forall t \geq 0 : S(t) \geq 0,\ F(0^-) = 0,$

$V(0^-,\theta) = 0, \forall \theta \geq 0, V(t,0) = (1-f)U'(t),$

• Individual cost $J_{indi}(\xi; V) = \mathbb{E}^{\xi}[g^V] = \langle \xi, g^V \rangle_{\mathbb{R}^{N+1}}$, g^V, $\xi \in \mathbb{R}^{N+1}$, to be minimized under the constraint ξ = (discrete) probability law.

• compatibility: when *all* individuals follow ξ vaccination is V.

Theorem (F. Salvarani, G.T. 2016)

The vaccination dynamics model admits a Nash equilibrium.

Summary for MFG vaccination models

• cost has the structure $\mathcal{C}(individual, societal)$; it is to be optimized with respect to the '*individual*' strategy, the '*societal*' remains fixed, i.e. $individual \mapsto \mathcal{C}(individual, societal)$;

• Nash / MFG equilibrium when '*individual*' is unilaterally optimal and '*societal*'= '*individual*' (similar to a fixed point);

• benevolent planner approach: minimize $individual \mapsto \mathcal{C}(individual, individual)$;

• in general neither a benevolent planner game (cost of anarchy ...), nor a zero-sum game.

Computing the equilibrium: semi-explicit schemes on metric spaces

• How to find and equilibrium ?
Example: (bi-linear, Hilbert) $\mathcal{C}(\eta,\xi) = \langle \eta, L\xi \rangle$ the dynamics $x'(\tau) = -Lx(\tau)$, i.e., $x(\tau) = e^{-L\tau}x_0$ goes to equilibrium.

• JKO goes to the minimum of $\langle \eta, L\eta \rangle = \frac{\langle \eta, L+L^T \eta \rangle}{2}$, i.e., the anti-symmetric part of L dissapears; in general obtain a ∇-flow converging to a "benevolent planner" perspective.

• semi-explicit numerical scheme for $\mathcal{C}(individual = \ \xi^I, global = \ \xi^G)$:
Algorithm: set $\xi_k = \xi_k^G = \xi_k^I$, and

$$\xi_{k+1} \in \mathrm{argmin}_{\eta \in \Sigma_{N+1}} \frac{dist(\eta, \xi_k)^2}{2\Delta\tau} + \mathcal{C}(\eta, \xi_k).$$

• related to "best reply" (MFG: cf. G. Carlier and co-workers) and "fictitious play" (MFG: cf. P. Cardialiaguet et al.) learning methods in game theory.

Computing MFG equilibrium

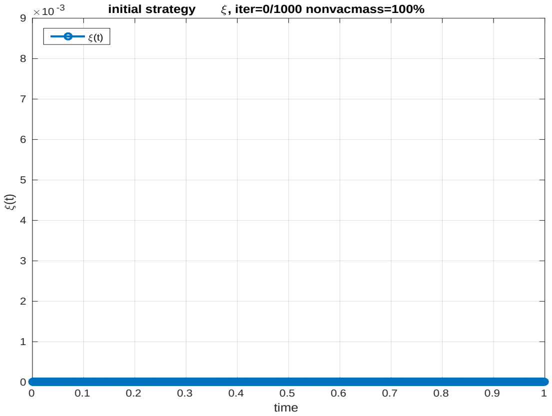
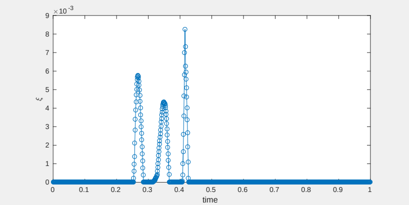
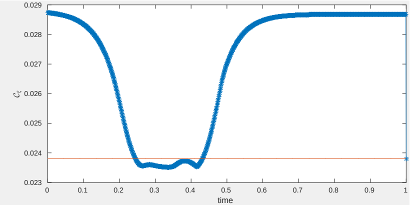


Figure: Notation: $\xi_\tau(t)$ is a time-dependent probability law over the possible vaccination times indexed by variable t. Initial data $\xi_{\tau=0}(t)$ (uniform) and iterations (VIDEO) of the vaccination MFG equilibrium strategy ξ_τ. Case: short persistence.

Short persistence, perfect efficacy





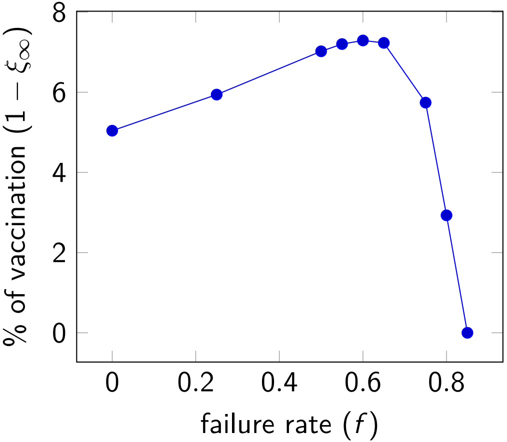
Solution strategy ξ^{MFG} supported at several time instants between 0.25 and 0.43, with 68% of non-vaccinating individuals.

Cost $\mathcal{C}_{\xi^{MFG}}$ of the optimal converged strategy ξ^{MFG}, red line: cost of the non-vaccinating pure strategy $(\mathcal{C}_{\xi^{MFG}})_{N+1}$

Effects of the failed vaccination rate on the strategy

Individual vaccination policy with respect to the failed vaccination rate of the vaccine.

f	$1 - \xi_\infty$
0	5.04
0.25	5.94
0.5	7.02
0.55	7.2
0.6	7.29
0.65	7.23
0.75	5.74
0.8	2.93
0.85	0



$r_V = 0.025$ (the other parameters are unchanged)
If $f = 0.85$ the probability of being infected is 14.38%.

MFG numerical schemes on metric spaces: theoretical results (GT '17)

- Question: explicit $\Delta\tau \to 0$ has a meaning ?

- Hilbert space: $\partial_\tau \xi(\tau,t) + \nabla_1 \mathcal{C}(\xi(\tau,t), \xi(\tau,t)) = 0$; metric space equivalent ?

Limit curve ? JKO strategy: compactness by equi-continuity and uniform boundedness.

- Upper bounds (assume: Lipschitz in the second, 'societal', strategy; relatively compact level sets ...):

$$\mathcal{C}(\xi_{k+1},\xi_{k+1}) \leq \mathcal{C}(\xi_{k+1},\xi_k) + Ld(\xi_{k+1},\xi_k) \leq \mathcal{C}(\xi_{k+1},\xi_k) + \frac{d(\xi_{k+1},\xi_k)^2}{2\Delta\tau} + \tau L^2/2 \leq \mathcal{C}(\xi_k,\xi_k) + \Delta\tau L^2/2 \leq ... \leq \mathcal{C}(\xi_0,\xi_0) + TL^2/2.$$

- equi-continuity: rework the details of the proof of ∇-flow of $\xi \mapsto \mathcal{C}(\xi,\nu)$
- CONCLUSION: when $\Delta\tau \to 0$ there exists a limit curve.

MFG numerical schemes on metric spaces: theoretical results (GT '17)

• give a meaning in a metric space to:
$\partial_\tau \xi(\tau,t) + \nabla_1 \mathcal{C}(\xi(\tau,t),\xi(\tau,t)) = 0$;
• literature: ∇-flows for $E(t,x)$: Ferreira-Valencia-Guevara '15, Rossi-Mielke-Savaré '08, C. Jun '12, Kopfer-Sturm '16

• EDI (pointwise) formulation (G.T. '17)
$\frac{d}{d\tau}\mathcal{C}(\xi_\tau,\nu)\Big|_{\nu=\xi_\tau} + \frac{1}{2}|\xi'_\tau|^2 + \frac{1}{2}|\nabla_1 \mathcal{C}|^2(\xi_\tau,\xi_\tau) \leq 0$ a.e.
does not use convexity but uses regularity hypothesis for $\mathcal{C}$.

• EVI formulation (G.T. '17)
$\mathcal{C}(\xi_\tau,\xi_\tau) + \frac{1}{2}\frac{d}{d\tau}d^2(\xi_\tau,y) + \frac{\lambda}{2}d^2(\xi_\tau,y) \leq \mathcal{C}(y,\xi_\tau), \forall y, \ a.e.\ \tau \geq 0.$
does not use much regularity uses λ-convexity.

• both are the limit of numerical schemes (under hyp.)

High order schemes for vaccination games: results by L. Laguzet

What about 2nd order schemes for MFG? Recent work by L. Laguzet, 3 high order schemes inspired from Heun, RK3, RK4:
The (standard, Hilbert space) Heun:

$$p_1 = x_k + \tau f(t_k, x_k),\ x_{k+1} = x_k + \frac{\tau}{2}\Big[f(t_k, x_k) + f(t_{k+1}, p_1) \Big].$$

The variational (metric space) Heun scheme

$$\widetilde{\xi}_{k+1} \in \underset{\eta \in \mathcal{A}}{\operatorname{argmin}} \left\{ \frac{d(\eta, \xi_k)^2}{2\tau} + \mathcal{C}(\eta, \xi_k) \right\}, \tag{7}$$

$$\xi_{k+1} \in \underset{\eta \in \mathcal{A}}{\operatorname{argmin}} \left\{ \frac{d(\eta, \xi_k)^2}{2\tau} + \frac{1}{2}\mathcal{C}(\eta, \xi_k) + \frac{1}{2}\mathcal{C}(\eta, \widetilde{\xi}_{k+1}) \right\}. \tag{8}$$

Two minimizations are required in order to obtain ξ_{k+1}.

High order schemes for MFG: numerical results (courtesy of L. Laguzet)

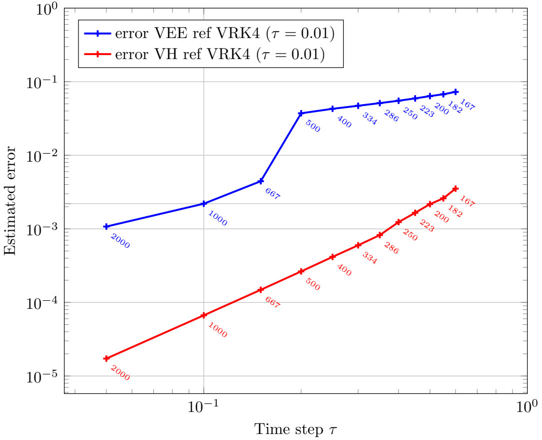


Figure: Explicit Euler (= "best reply", noted VEE) and Variational Heun (VH) schemes.

Outline

Perspectives

- more examples for ∇-flows
- general proof of the order of convergence of VIM, EVIE
- general evolution on metric spaces, not only ∇-flows
- other higher order schemes (for non-MFG settings)
- MFG: does evolution converges to an equilibrium ?
- MFG: impose some evolution for the global variable ?
- ...