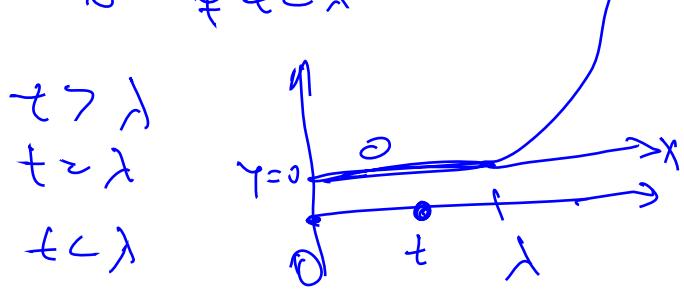


$$\underline{\text{Ex 2.4 page 24}} \quad \stackrel{1^{\circ}}{=} \quad x_{\lambda}(t) \approx \begin{cases} (t-\lambda)^2 & t > \lambda \\ 0 & t = \lambda \\ 0 & t < \lambda \end{cases}$$

Bruni

$$\frac{dx_{\lambda}(t)}{dt} \Rightarrow \begin{cases} z(t-\lambda) & t > \lambda \\ 0 & t = \lambda \\ 0 & t < \lambda \end{cases}$$



For $t=\lambda$ we check that left/right derivatives are equal $\rightarrow 0.$ ✓

$$x'(t) = f(t, x(t)) \quad f(t, 0) \approx 2\sqrt{|0|} \quad \text{not Lipschitz.}$$

Thus $x'_{\lambda}(t) = \frac{1}{t-\lambda} \cdot 2(t-\lambda) = 2|x(t)|^{1/2}$ so $x_{\lambda}(t)$

is always a solution ($\forall \lambda \in \mathbb{R}, \infty \cup \{-\infty\}$ no eigenvalues)

$$x_0(t) = f_2 \quad x_{\infty}(t) = 0$$

$\stackrel{2^0}{\equiv}$ $x_{n+1} = x_n + h \underbrace{f}_{f(t_n, x_n)}$

$$x_0 = 0$$

$$x_1 = x_0 + h \cdot 2\sqrt{|x_0|} = 0 + 2 \cdot h \sqrt{|0|} = 0$$

$$x_2 = x_1 + h \cdot 2\sqrt{|x_1|} = 0 + 2 \cdot h \cdot \sqrt{|0|} = 0$$

...
 $x_n = 0 \quad \forall n$ so solution found is $x_{\lambda}(t)$ with $\lambda = \infty.$

Case 1 infinite precision computations

Case 2 finite precision computations $\varepsilon = 10^{-16}, \quad h \approx 10^{-3} \dots 10^{-6}$

$$x_0 = 0 + \varepsilon \approx 10^{-16}$$

$$x_1 = x_0 + 2h\sqrt{|x_0|} \approx 10^{-16} + 2h\sqrt{10^{-16}} \approx 2h \cdot 10^{-8} \approx [10^{-14}, 10^{-9}]$$

$$x_2 = x_1 + 2h\sqrt{|x_1|} \approx 10^{-16} + 2h \cdot [10^{-7}, 10^{-15}] \approx [10^{-13}, 10^{-6}]$$

$$x_3 = \dots \quad [10^{-4}, 10^{-12}] > 0$$

On the other side the EE scheme will converge (cf. course) to a solution $x_{\lambda}(t);$ therefore $\lambda = 0. \quad (t^2)$

EI (Implicit Euler)

$$x_{n+1} = x_n + h \frac{f_{n+1}}{f(t_{n+1}, x_{n+1})}$$

$$f(t_{n+1}, x_{n+1})$$

For (2.43) we obtain

$$x_0 = 0 \quad (\text{+ error?})$$

$x_1 = x_0 + h \cdot 2 \cdot \sqrt{|x_1|} \Rightarrow x_1 = \text{the square of the solution of } y \approx \sqrt{|x_1|} \geq 0.$

the equation $y^2 - 2hy - x_0 = 0$

$$y_{1,2} = \frac{2h \pm \sqrt{4h^2 + 4x_0}}{2} = h \pm \sqrt{h^2 + x_0}$$

e.g. has 2 solutions
their sum is $2h$
their product $-x_0$.

If $x_0 > 0$ there is only one positive solution if $\sqrt{h^2 + x_0} \approx O(h) > 0$

If $x_0 < 0$ there are 2 positive solutions: $h + \sqrt{h^2 + x_0} > 0$
 $h - \sqrt{h^2 + x_0} \approx 0 \dots$

$$\frac{h + \sqrt{h^2 + x_0}}{h^2 - (h^2 + x_0)} \approx -\frac{2h}{x_0} \approx$$

Sign of roots $h + \sqrt{h^2 + x_0} > 0$ since $x_0 \ll h$
 $h - \sqrt{h^2 + x_0} \geq 0?$ ($\Rightarrow h \geq \sqrt{h^2 + x_0} \Leftrightarrow h^2 \geq h^2 + x_0$)

Which one is the "good" EI scheme value? BOTH! There is a user choice!

Let us continue: $x_2 = x_1 + 2h \sqrt{|x_2|}$. This case is similar to the one before when $x_1 \approx 0$ we can have 2 positive solns and have to choose one!

This goes on... which means that in practice by our choice we can obtain any solution $x_N(t)$



Often in practice we obtain $x_{N=0}(t) \approx t^2$.

$\varepsilon \geq 5$ p 24 Point 1 we want to solve $y = x + h\psi(t, x, y)$

Picard iterations $y_0 = x$ $y_{n+1} = x + h\psi(t, x, y_n)$
Find thru "if & contraction (y_n) c.v" $y_{n+1} = F(y_n)$

Let us prove that $F(\underline{y})$ is a contraction, that is

$$F(z_1) - F(z_2) \leq C |z_1 - z_2| \text{ with } C < 1.$$

$$\begin{aligned} F(z_1) - F(z_2) &= x + h\psi(t, x, z_1) - \\ (x + h\psi(t, x, z_2)) &= h \left[\psi(t, x, z_1) - \psi(t, x, z_2) \right] \\ &\leq h L_y |z_1 - z_2| \end{aligned} \quad \boxed{F(\underline{y}) = x + h\psi(t, x, \underline{y})}$$

Thus when $h < \frac{1}{L_y}$ the function F is a contraction.

thus by the Picard thm. $y_n \rightarrow y$. But $y_{n+1} = \underbrace{x + h\psi(t, x, y_n)}_{\downarrow}$ follows
 by passing at the limit ($n \rightarrow \infty$) we have: $y = x + h\psi(t, x, y)$

thus the limit is a solution of (2.45).

To prove uniqueness suppose y, \tilde{y} are two solutions. Then $y \neq \tilde{y}$

$$\begin{aligned} y = F(y) \quad \Rightarrow \quad |y - \tilde{y}| &= |F(y) - F(\tilde{y})| \leq h L_y |y - \tilde{y}|, \text{ absurd because} \\ \tilde{y} = F(\tilde{y}) \quad \Rightarrow \quad h L_y &\leq 1. \text{ Thus } y = \tilde{y} \Rightarrow \text{ solution is unique.} \end{aligned}$$

For any $t|x, h$ we denote the solution $y \in S(t, x, h)$.

2° $y = x + h\phi(t, x)$. We want to prove that ϕ is Lipschitz. $y = S(t, x, h)$. $\phi(t, x) = \frac{y - x}{h} = \frac{s(t, x, h) - x}{h}$ {for small h }

$h > 0$, cst. To prove that ϕ is Lipschitz is enough to prove s is Lipschitz. Recall s is the solution of $\dot{y} = x + h\psi(t, x, y)$

We have to prove $|s(t, x, h) - s(t, \tilde{x}, h)| \leq L_s |x - \tilde{x}| \forall x, \tilde{x}$.

$$\begin{cases} \dot{y} = x + h\psi(t, x, y) \\ \dot{\tilde{y}} = \tilde{x} + h\psi(t, \tilde{x}, \tilde{y}) \end{cases} \quad |y - \tilde{y}| = |x + h\psi(t, x, y) - (\tilde{x} + h\psi(t, \tilde{x}, \tilde{y}))| \leq \\ |x - \tilde{x}| + h |\psi(t, x, y) - \psi(t, \tilde{x}, \tilde{y})| \leq |x - \tilde{x}| + h \cdot [|\psi(t, x, y) - \psi(t, x, \tilde{y})| + |\psi(t, \tilde{x}, \tilde{y}) - \psi(t, x, \tilde{y})|] \\ \leq |x - \tilde{x}| \cdot (1 + L_{\psi, x}) + |y - \tilde{y}| \cdot L_{\psi, y} \end{math>$$

