Consistance R-K (page 17 poly Thm 2.14) TD4

Compute the truncation error $\tau_{n+1}(h)$

$\tau_{n+1}(h) = \frac{X_{n+1} - U^{*}_{n+1}}{h}$ ← obtained from the R-K scheme starting from $U_n = X(t_n) = X_n$

$$\tau_{n+1}(h) = \frac{X_{n+1} - \left\{ X_n + h \sum_{i=1}^{s} b_i f[t_n + O(h), X_n + O(h)] \right\}}{h}$$

$$= \frac{X_{n+1} - X_n - h \sum b_i [f(t_n, X_n) + O(h)]}{h} \qquad B = \sum_{i=1}^{s} b_i$$

$$= \underbrace{\frac{X_{n+1} - X_n - h B f(t_n, X_n)}{h}} - B \cdot O(h)$$

Since $f(t_n, X_n) = X'(t_n)$ $\quad \tau_{n+1}(h) = \frac{X(t_n + h) - X(t_n) - h X'(t_n) \cdot B}{h} - B\, O(h)$

But $\frac{X(t_n+h) - X(t_n) - X'(t_n) \cdot h}{h} = \frac{O(h^2)}{h} = O(h)$

So $\tau_{n+1}(h) = O(h) + \underbrace{\frac{h(1-B) f(t_n, X_n)}{h}} - B\, O(h) = O(h) + \underbrace{(1-B) f_n}$

For consistancy $\tau(h)$ has to $\to 0$ when $h \to 0$. So since f_n is not cv. to zero we have to have $B = 1$.

Ex 2.14 page 26. 1°) We check $\sum_{k=0}^{s} a_k k = \sum_{k=0}^{s} b_k$

BDF-2 $\quad y_{n+2} - \frac{4}{3} y_{n+1} + \frac{1}{3} y_n = \frac{2}{3} h f(t_{n+2}, y_{n+2})$

$a_2 = 1 \quad a_1 = -\frac{4}{3} \quad a_0 = \frac{1}{3} \qquad b_0 = b_1 = 0 \quad b_2 = \frac{2}{3}$

$$\sum_{k=0}^{s} k a_k = 0 \cdot \frac{1}{3} + 1 \cdot \left(-\frac{4}{3}\right) + 2 \cdot 1 = \frac{2}{3} = 0 + 0 + \frac{2}{3} = \sum_{k=0}^{s} b_k.$$

$\underline{2^\circ}$ Truncation error for BDF-2

$$\tau_{n+2}(h) = \frac{1}{h}\left\{ \sum_{k=0}^{s} X(t_{n+k})\, a_k - \sum_{k=0}^{s} h\, b_k\, f(t_{n+k}, X_{n+k})\right\}$$

$$= \frac{1}{h}\left\{ X_{n+2} - \frac{4}{3}X_{n+1} + \frac{1}{3}X_n - h\cdot\frac{2}{3}\underbrace{f(t_{n+2}, X_{n+2})}_{X'(t_{n+2})}\right\}$$

Taylor expansions around $\cancel{t_n}$, $\cancel{t_{n+1}}$, t_{n+2}.

$$X_{n+1} = X(t_{n+1}) = X(t_{n+2} - h) = X(t_{n+2}) + X'(t_{n+2})\cdot(-h) + X''(t_{n+2})\cdot\frac{h^2}{2} + O(h^3).$$

$$X_n = X(t_{n+2} - 2h) = X_{n+2} + X'(t_{n+2})\cdot(-2h) + X''(t_{n+2})\cdot 2h^2 + O(h^3)$$

Thus $\tau_{n+2}(h) = \frac{1}{h}\left\{ X_{n+2}\cdot\underbrace{\left(1 - \frac{4}{3} + \frac{1}{3}\right)}_{0} + X'(t_{n+2})\underbrace{\left(\frac{4}{3}\cdot h - \frac{2}{3}h - \frac{2}{3}h\right)}_{0}\right.$

$$\left. + X''(t_{n+2})\underbrace{\left(-\frac{2}{3}h^2 + \frac{1}{3}\cdot 2h^2\right)}_{0} + O(h^3)\right\} = \underline{\underline{O(h^2)}}$$

So the truncation error is at least 2^{nd} order. Let us see whether it is of higher order. The term of order 2 is $-\frac{4}{3}\cdot X^{(3)}(t_{n+2})\frac{(-h)^3}{6}$

$$+ \frac{1}{3}\cdot X^{(3)}(t_{n+2})\frac{(-2h)^3}{6} = \frac{X^{(3)}(t_{n+2})h^3}{6}\underbrace{\left\{\frac{4}{3} - \frac{8}{3}\right\}}_{\neq 0}.$$ So $\tau_{n+2}(h)$ is not of order 3.

3°) $\underline{\text{Adam-Bashford}}$ $\quad Y_{n+2} = Y_{n+1} + \frac{3}{2}hf(t_{n+1}, Y_{n+1}) - \frac{1}{2}hf(t_n, Y_n)$

$$\tau_{n+2}(h) = \frac{1}{h}\left\{ X_{n+2} - X_{n+1} - \frac{3}{2}f(t_{n+1}, X_{n+1})\cdot h + \frac{1}{2}h f_n\right\}$$

Taylor expansion around t_{n+1}. $X_{n+2} = X(t_{n+1} + h) = X_{n+1} + X'(t_{n+1})\cdot h + X''(t_{n+1})\cdot h^2/2 + O(h^3)$

$$f(t_n, x_n) = x'(t_n) = x'(t_{n+1} - h) = x'(t_{n+1}) - h\, x''(t_{n+1}) + O(h^2)$$

$$\tau_{n+2}(h) = \frac{1}{h}\Big\{ \underbrace{x_{n+1} + h\, x'_{n+1} + \frac{h^2}{2} x''_{n+1} + O(h^3)}_{x_{n+2}} - x_{n+1} - \frac{3}{2} x'_{n+1} \cdot h$$

$$+ \frac{1}{2} \cdot h \left[x'_{n+1} - h\, x''_{n+1} + O(h^2) \right] \Big\}$$

$$= \frac{1}{h} \Big\{ x_{n+1} \underbrace{(1 - 1)}_{0} + x'_{n+1} \cdot h \underbrace{\left(1 - \frac{3}{2} + \frac{1}{2}\right)}_{0} + x''_{n+1} h^2 \underbrace{\left(\frac{1}{2} - \frac{1}{2}\right)}_{0}$$

$$+ O(h^3) \Big\} = O(h^2).$$

So $\tau_{n+2}(h)$ is at least of order 2.. To check whether it is of superior order we compute the coeff of 2^{nd} order. It is

$$\frac{h^3}{6} x^{(3)}_{n+1} + \frac{1}{2} \cdot h\, x^{(3)}_{n+1} \cdot \frac{h^2}{2} = x^{(3)}_{n+1} h^3 \left(\frac{1}{6} + \frac{1}{4} \right) \neq 0.$$

<u>Ex 2.10</u> EI for SIR

$$\begin{cases} \dot S = -\beta S I \\ \dot I = \beta S I - \gamma I \\ \dot R = \gamma I \end{cases}$$

$U^S \simeq S$ $\quad U^R \simeq R$

$U^I \simeq I$

<u>EI</u>

$$\begin{pmatrix} U^S_{n+1} \\ U^I_{n+1} \\ U^R_{n+1} \end{pmatrix} = \begin{pmatrix} U^S_n \\ U^I_n \\ U^R_n \end{pmatrix} + h \underbrace{\begin{pmatrix} -\beta U^S_{n+1} U^I_{n+1} \\ \beta U^S_{n+1} U^I_{n+1} - \gamma U^I_{n+1} \\ \gamma U^I_{n+1} \end{pmatrix}}_{\begin{pmatrix} -\beta U^I_{n+1} & 0 & 0 \\ 0 & (\beta U^S_{n+1} - \gamma) & 0 \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} U^S_{n+1} \\ U^I_{n+1} \\ U^R_{n+1} \end{pmatrix}}$$

This is not a linear system to solve!

<u>EE</u>

$$\begin{pmatrix} U^S_{n+1} \\ U^I_{n+1} \\ U^R_{n+1} \end{pmatrix} = \begin{pmatrix} U^S_n \\ U^I_n \\ U^R_n \end{pmatrix} + h \begin{pmatrix} -\beta U^S_n U^I_n \\ (\beta U^S_n - \gamma) U^I_n \\ \gamma U^I_n \end{pmatrix}$$