Ex 3.1 page 58 1° Show that

$$f'(x) = \frac{2f(x+3h) - 9f(x+2h) + 18f(x+h) - 11f(x)}{6h} + O(h^3)$$

Taylor expansion of the RHS :

$$f(x+3h) = f(x) + 3h f'(x) + \frac{9h^2}{2} f^{(2)}(x) + \frac{(3h)^3}{6} f^{(3)}(x) + O(h^4)$$

$$f(x+2h) = f(x) + 2h f'(x) + \frac{4h^2}{2} f^{(2)}(x) + \frac{8h^3}{6} f^{(3)}(x) + O(h^4)$$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f^{(2)}(x) + \frac{h^3}{6} f^{(3)}(x) + O(h^4)$$

$$\text{Thus (RHS)} = \frac{1}{6h}\Big\{ f(x)\big(\underbrace{2-9+18-11}_{0}\big) + f'(x)\cdot h\big(\underbrace{2\cdot 3 - 9\cdot 2 + 18\cdot 1}_{6}\big) + f^{(2)}(x)h^2\Big(\underbrace{2\cdot\frac{9}{2} - 9\cdot\frac{4}{2} + 18\cdot\frac{1}{2}}_{0}\Big) + f^{(3)}(x)h^3\Big(\underbrace{\frac{2\cdot 27}{6} - 9\cdot\frac{8}{6} + 18\cdot\frac{1}{6}}_{0}\Big)\Big\} + O(h^3) = \frac{1}{6h}\Big\{f'(x)\cdot 6h + O(h^4)\Big\} = f'(x) + O(h^3)$$

$+O(h^4)$

2° Find $\alpha, \beta, \gamma, \delta$ such that

$$f'(x) = \frac{\alpha f(x+2h) + \beta f(x+h) + \gamma f(x) + \delta f(x-h)}{h} + O(h^3)$$

We use the Taylor expansions for $f(x+2h)$, $f(x+h)$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f^{(2)}(x) - \frac{h^3}{6} f^{(3)}(x) + O(h^4)$$

$$\text{(RHS)} = \frac{1}{h}\Big\{ f(x)\big(\underbrace{\alpha+\beta+\gamma+\delta}_{0}\big) + h f'(x)\big(\underbrace{2\alpha+\beta-\delta}_{1}\big) + h^2 f^{(2)}(x)\Big(\underbrace{2\alpha + \frac{\beta}{2} + \frac{\delta}{2}}_{0}\Big) + h^3 f^{(3)}(x)\Big(\underbrace{\frac{8}{6}\alpha + \frac{\beta}{6} - \frac{\delta}{6}}_{0}\Big) + O(h^4)\Big\}$$

In order for the (RHS) to be equal to $f'(x) + O(h^3)$ we need

$\alpha+\beta+\gamma+\delta = 0$, $2\alpha+\beta-\delta = 1$, $2\alpha + \beta/2 + \delta/2 = 0$, $\frac{4}{3}\alpha + \frac{\beta-\delta}{6} = 0$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 2 & 1/2 & 0 & 1/2 \\ 4/3 & 1/6 & 0 & -1/6 \end{pmatrix}}_{A=}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow \dots$$

$$\det A = 1\cdot\begin{vmatrix} 2 & 1 & -1 \\ 2 & 1/2 & 1/2 \\ 4/3 & 1/6 & -1/6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 2 & 1/2 & 1 \\ 4/3 & 1/6 & 0 \end{vmatrix} = \frac{2}{6} - \frac{4}{3} \neq 0$$

Rq $f(x+h)+f(x-h)$ = symmetric around $h=0$

$$= 2f(x)+\underbrace{0}_{=(h-h)}\cdot f'(x)+h^2 f^{(2)}(x)+\underbrace{\left(\frac{h^3}{6}-\frac{h^3}{6}\right)f^{(3)}(x)}_{0}+\dots$$

$f(x+h)-f(x-h)$ = antisymmetric w/r to $h=0$

$$0\cdot f(x)+f'(x)\cdot 2+0\cdot f^{(2)}(x)+\dots \qquad 0\cdot f^{(4)}(x)+\dots$$

Rq Similar formulas exist for approximating $f''(x)$, $f'''(x)$....

Ex
$$\boxed{f''(x)=\frac{f(x+h)+f(x-h)-2f(x)}{h^2}+O(h^2)}$$

- and for functions of several variables $f(x,y)$ $\frac{\partial^2 f}{\partial x^2}$ ✓ $\frac{\partial^2}{\partial x\partial y}f(x,y)$

Ex examination 2015 $u_{n+1}=u_n+h\left(\frac{k_1}{6}+\frac{2k_2}{3}+\frac{k_3}{6}\right)$

$k_1=f_n$, $k_2=f(t_n+\frac{h}{2}, u_n+h\frac{k_1}{2})$ $k_3=f(t_n+h, u_n-hk_1+2hk_2)$

1) Write the Butcher table for this numerical (RK) scheme.

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0\\ 1/2 & 1/2 & 0 & 0\\ 1 & -1 & 2 & 0\\ \hline & 1/6 & 2/3 & 1/6\end{array}$$

$c_1=0$ ($k_1 = f(t_n+0\cdot h,\dots)$) $c_2=1/2$ (because of k_2)

$c_3=1$ (because of k_3)

$b_1=1/6$ (coeff before k_1), $b_2=2/3$ (k_2), $b_3=1/6$ (k_3)

$a_{11}=0=a_{12}=a_{13}$ (because of k_1) Scheme is explicit thus

$a_{11}=a_{12}=a_{13}=a_{22}=a_{23}=a_{33}=0$.

$a_{21}=1/2$ (because of k_2) $a_{31}=-1$ (because of k_3)

$a_{32}=2$ (because of k_3)

2) Remind/Recall why the scheme is consistent.
The scheme is consistent iff $\sum_{i=1}^{s} b_i=1$ (OK here because $\frac{1}{6}+\frac{2}{3}+\frac{1}{6}=1$)

3 | h Find the stability region Take $f(x) = \lambda x$

$k_1 = \lambda U_n \quad k_2 = f(t_n + h/2, u_n + h\frac{k_1}{2}) = \lambda(U_n + \frac{\lambda h U_n}{2}) = \lambda U_n + \frac{\lambda^2 h}{2} U_n$

$k_3 = \lambda\left[U_n - h\lambda U_n + 2h\left(\lambda U_n + \frac{\lambda^2 h}{2} U_n\right)\right]$

Denote $z = h\lambda$ Then $U_{n+1} = U_n + \frac{1}{6} h \cdot \underbrace{\lambda U_n}_{k_1} + \frac{2}{3} h \left(\lambda U_n + \frac{\lambda^2 h}{2} U_n\right)$

$+ \frac{1}{6} h \cdot \lambda U_n \left[1 - h\lambda + 2h\left(\lambda + \frac{\lambda^2 h}{2}\right)\right]$

$$U_{n+1} = U_n + \frac{1}{6} z U_n + \frac{2}{3} z U_n \left(1 + \frac{z}{2}\right) + \frac{1}{6} \cdot z U_n \left[\underbrace{1 - z + 2z\left(1 + \frac{z}{2}\right)}_{1 + z + z^2}\right]$$

$$U_{n+1} = U_n \left[1 + \frac{z}{6} + \frac{2}{3} z + \frac{z^2}{3} + \frac{1}{6} z + \frac{z^2}{6} + \frac{z^3}{6}\right]$$

$$U_{n+1} = U_n \left\{1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right\}.$$

Stability region $\left\{ z \in \mathbb{C} \;\middle|\; \left|1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right| < 1 \right\}$.