EDS exercises

Ex 4.4. 1 $Y_n \sim X(t_n)$ $\quad dX = a\,dt + b\,dW_t$

$a = a(t, X) \ldots$

$$Y_{n+1} = Y_n + a_n h + b_n \underbrace{\xi_n \sqrt{h}}$$

random variable with mean: $\sqrt{h}\left[\frac{1}{2}(1) + \frac{1}{2}(-1)\right] = 0$

and variance $= \left(\frac{1}{2}\cdot 1^2 + \frac{1}{2} 1^2\right)\cdot h = h$

$\xi_n \sqrt{h}$ has same mean and variance as $W_{t+h} - W_t$

let us check the weak consistency conditions:

(W1) $\mathbb{E}\left(\frac{Y_{n+1} - Y_n}{h} \Big| \mathcal{A}_{t_n}\right) - a_n = \mathbb{E}\left[\frac{a_n h + b_n \sqrt{h}\,\xi_n}{h} \Big| \mathcal{A}_{t_n}\right]$

$-a_n = a_n + b_n \cdot \underbrace{\mathbb{E}\left[\frac{\xi_n}{\sqrt{h}} \Big| \mathcal{A}_{t_n}\right]}_{=0 \text{ since } \xi_n \perp\!\!\!\perp \mathcal{A}_{t_n} \text{ and } \mathbb{E}(\xi_n) = 0} - a_n = 0.$ OK

(W2) $\underbrace{\mathbb{E}\left[\frac{(Y_{n+1} - Y_n)^2}{h} \Big| \mathcal{A}_{t_n}\right]}_{B} = \mathbb{E}\left\{\frac{(a_n h + b_n \xi_n \sqrt{h})^2}{h} \Big| \mathcal{A}_{t_n}\right\}$

$= a_n^2 \cdot h + a_n \cdot b_n \sqrt{h}\, \mathbb{E}(\xi_n) + b_n^2 \frac{(\sqrt{h})^2}{h} \mathbb{E}(\xi_n^2)$

$= a_n^2 \cdot h + b_n^2$ $\quad$ Thus $|(B) - b_n^2| = a_n^2 \cdot h \xrightarrow[h \to 0]{L^2} 0.$

(we supposed a, a', b, b' bounded).

(2) The same works for any other variables indep., with mean $= 0$, variance $= 1$.

(3) Strong consistence? The first condition is the same so we look at the second condition; the important part is $\frac{1}{h}\left| Y_{n+1} - Y_n - \underbrace{\mathbb{E}[Y_{n+1} - Y_n / \mathcal{F}_{\tau_n}]} - b_n \Delta W_n \right|$

$$= \frac{1}{h}\left| a_n \cancel{h} + b_n \sqrt{h}\, \xi_n - a_n \cancel{h} - b_n \Delta W_n \right| = \frac{|b_n|}{\sqrt{h}} \cdot \left| \xi_n - \frac{\Delta W_n}{\sqrt{h}} \right|$$

The term $\Big| \xi_n - \underbrace{\frac{\Delta W_n}{\sqrt{h}}}_{\mathcal{N}(0,1)} \Big|$ has a law independent of h so is small ($h \to 0$) only when it is zero i.e. law of $\xi_n =$ normal law; in fact we need $\xi_n = \frac{\Delta W_n}{\sqrt{h}}$.

Ex 4.6 we use Taylor formulas for a and b

$$a(Y_n + a_n h + b_n \Delta W_n) = \underbrace{a(Y_n)}_{\text{not: } a_n} + \underbrace{a'(Y_n)}_{a_n'} \cdot (a_n h + b_n \Delta W_n)$$

$$+ \frac{a''(\alpha_n^a)}{2} \cdot (a_n h + b_n \Delta W_n)^2 \quad \text{and same for } b.$$

$$\mathbb{E}\left[\frac{Y_{n+1} - Y_n}{h} \Big| \mathcal{F}_{\tau_n}\right] - a_n = \frac{a_n}{2} + \frac{1}{2}\Big\{ a_n + a_n' (a_n h + b_n \mathbb{E}[\Delta W_n])$$

$$+ \mathbb{E}\Big[\frac{a''}{2}(\cdot)\,(a_n h + b_n \Delta W_n)^2 \Big| \mathcal{F}_{\tau_n}\Big]\Big\} - a_n + \frac{1}{2}\, b_n \cdot \frac{\mathbb{E}(\Delta W_n)}{h} +$$

$$+\frac{1}{2}\,\mathbb{E}\Big\{\big(\underbrace{b_n + b_n'\cdot(a_nh + b_n\Delta W_n) + \frac{b''(\cdot)}{2}(a_nh+b_n\Delta W_n)^2}\big)$$

($\cdot$ stands for $\underline{b(Y_n + a_nh + b_n\Delta W_n)}$)

$$\cdot\,\frac{\Delta W_n}{h}\,\Big|\,\mathcal{F}_{\tau_n}\Big] = \frac{1}{2}\cdot\frac{b_n'\cdot b_n}{h}\,\underbrace{\mathbb{E}[(\Delta W_n)^2\,|\,\mathcal{F}_{\tau_n}]}_{=\mathbb{E}(\Delta W_n)^2 = h}$$

$$+\ \mathcal{O}(\sqrt{h}) = \frac{b_n b_n'}{2} + \mathcal{O}(\sqrt{h})$$

<u>BUT</u> for consistency this term has to $\to 0$ ($h\to 0$)

thus we need $\frac{b_n b_n'}{2} = 0 \Rightarrow$ either $b\equiv 0$ or $b'\equiv 0$

Thus first condition is only valid when $b=$ cst

$b(Y) = b_0 =$ constant.

This is a necessary condition. To prove it is sufficient we need to check 2nd consistency condition.

We obtained $\mathbb{E}\Big\{\frac{Y_{n+1}-Y_n}{h}\,\Big|\,\mathcal{F}_{\tau_n}\Big\} = a_n + \frac{a_n'a_n}{2}h$

$$+\,\mathbb{E}\Big\{\frac{a_n''(\cdot)}{2}\cdot(a_nh+b_n\Delta W_n)^2\,\Big|\,\mathcal{F}_{\tau_n}\Big\}.$$

Then $\frac{1}{h}\Big|Y_{n+1}-Y_n-\mathbb{E}\{Y_{n+1}-Y_n\,|\,\mathcal{F}_{\tau_n}\} - b_n\Delta W_n\Big|$

$$=\frac{1}{h}\Big|h\cdot\underline{\frac{a_n + a_n(Y_n + a_nh + b_n\Delta W_n)}{2}} + \cancel{b_n\Delta W_n} - \cancel{b_n\Delta W_n} - a_nh -$$

$$\frac{a_na_n'}{2}h^2 - \mathbb{E}\Big[\frac{h}{2}\,a_n''(\cdot)\,(a_nh+\Delta W_n\cdot b_n)^2\,\big|\,\mathcal{F}_{\tau_n}\Big]\Big|$$

$$= \frac{\Theta(h^2)}{h} + \frac{1}{h}\left| \frac{a(Y_n + a_n h + b_n \Delta W_n) - a(Y_n)}{2} \right| \cdot h$$

$$= \Theta(h) + \frac{1}{2}\Big| a'(Y_n)\cdot \underbrace{(a_n h + b_n \Delta W_n)}_{\Theta(\sqrt{h}) \to 0} + \underbrace{\frac{a''(\cdot)}{2}(a_n h + b_n \Delta W_n)^2}_{\Theta(h) \to 0} \Big|$$

$\to \Theta(h \to 0)$, thus condition is satisfied.