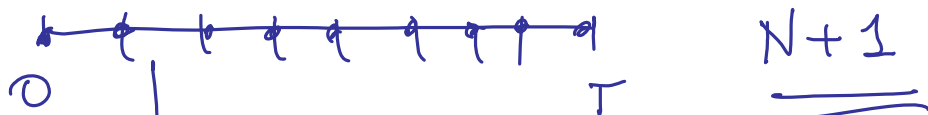


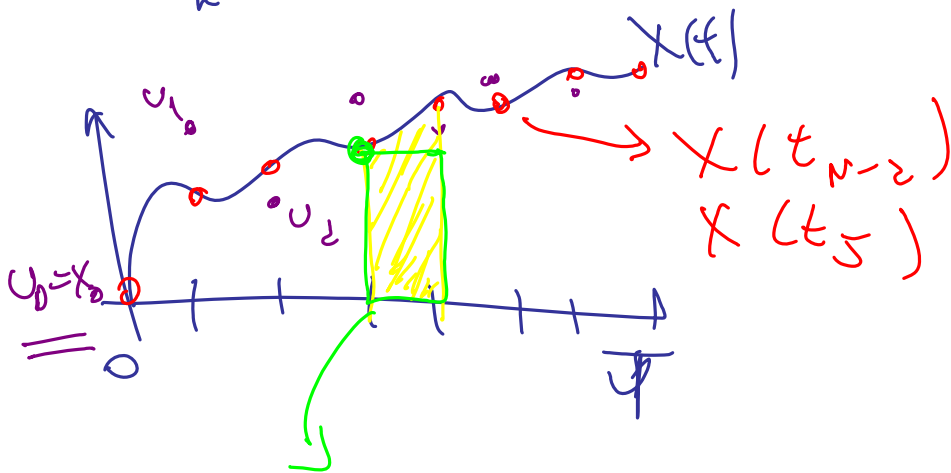
$t=0$ $t=1$ N points



$$h = T/N$$

$$t_n = nh$$

$X(t_n)$

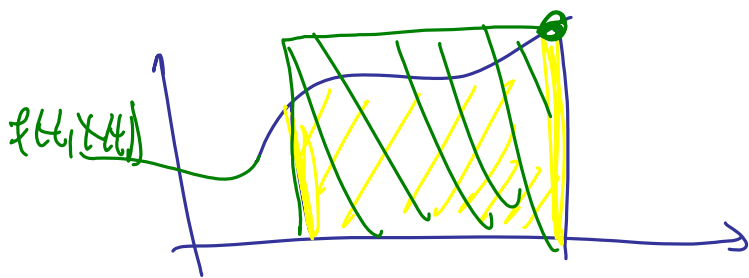


$$U_{n+1} = U_n + h \underbrace{f(t_n, U_n)}_{f_n}$$

Euler Explicit

$$f(t, u) = r \cdot u^2$$

$$U_{n+1} = U_n + h f(t_n, U_n) = U_n + h r \cdot U_n^2$$



E Implicit

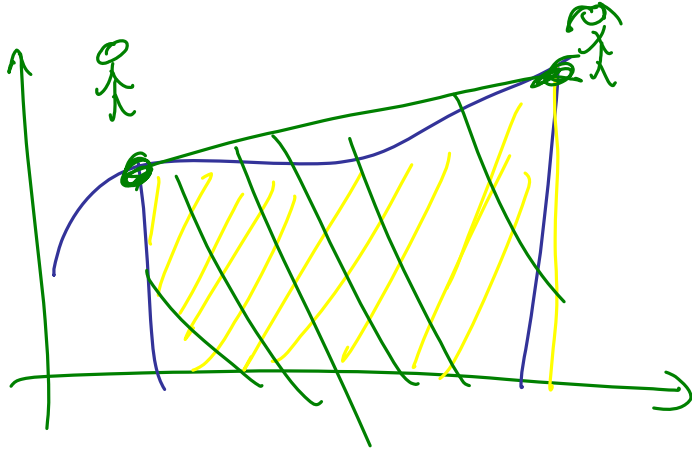
$$U_{n+1} = U_n + h f(t_{n+1}, U_{n+1})$$

$$U_{n+1} = U_n + h \cdot r U_{n+1}^2$$

$$f(r, u) = \frac{\tanh(u)}{1 - u^2}$$

$$1 - u^2$$

$$U_{n+1} = U_n + h \cdot \frac{\tanh(U_{n+1})}{1 - U_{n+1}^2}$$



$$\frac{\text{stick figure} + \text{stick figure}}{2} = h$$

$$U_n = U_0 + \frac{r h}{\Sigma} (U_0 + U_1) \Rightarrow$$

$$U_1 = \frac{1 + \frac{r h}{\Sigma}}{1 - \frac{r h}{\Sigma}} U_0$$

$$U_2 = U_1 + \frac{r h}{\Sigma} (U_1 + U_2) \Rightarrow$$

$$U_2 = \frac{1 + \frac{r h}{\Sigma}}{1 - \frac{r h}{\Sigma}} U_1$$

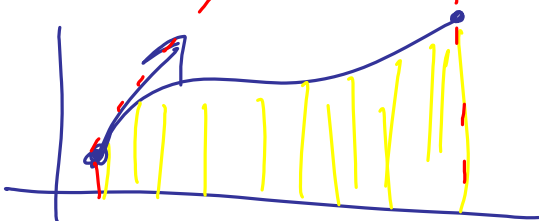
$$U_{na}^{\Sigma\Sigma} = U_n + h \cdot \underbrace{f(t_n, U_n)}_{\neq f(t_{na}, U_{na})}$$

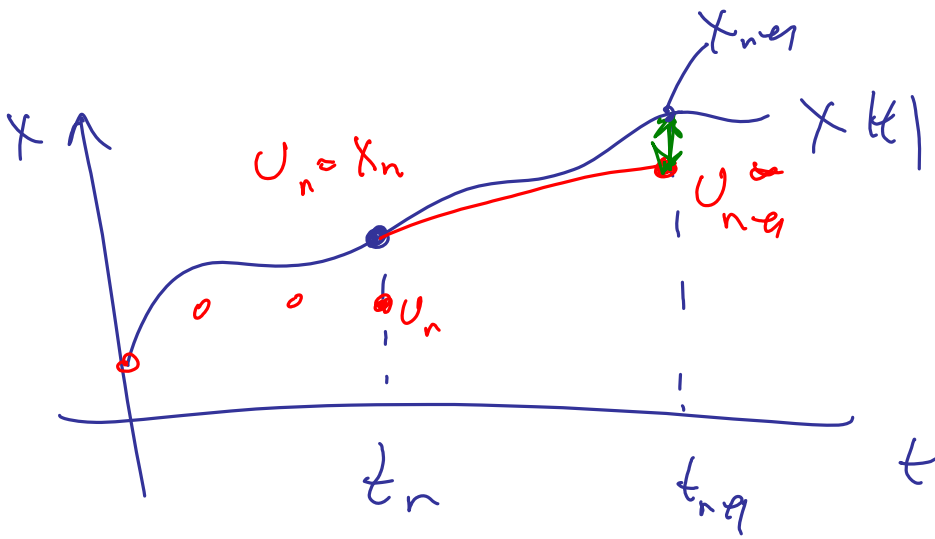
$U_{na}^{\Sigma\Sigma}$
 $\approx U_n^{\Sigma\Sigma} + h \cdot f(t_n, U_n^{\Sigma\Sigma})$

Mean pour $t_2(t_1, \text{stick figure}) = r \cdot \text{stick figure}^2$

$$U_{na} = U_n + \frac{h}{\Sigma} \left(r U_n^2 + r \cdot \underbrace{(U_n + h r U_n^2)}_{f_n} \right)$$

$$U_{na} = U_n + \frac{h}{\Sigma} \cdot \left[r U_n^2 + r (U_n + h r U_n^2) \right]$$





$$U_n = x_n : \quad U_{n+1}^{\text{ex}} = U_n + h \phi(t_n, U_n, \dots)$$

$$U_{n+1}^{\text{ex}} = x_{n+1} + h \phi(t_n, x_n, \dots)$$

$$\tau_{\text{trunc}}(h) = \frac{x_{n+1} - U_{n+1}^{\text{ex}}}{h}$$

Error truncature \mathcal{E}_T

$$\tau_{\text{trunc}}(h) = \frac{x_{n+1} - x_n - h \phi^{\mathcal{E}_T}(\dots)}{h} = \frac{x(t_{n+1}) - x(t_n) - h \phi(t_n, x_n)}{h}$$

$$= \frac{x(t_{n+1}) - x(t_n) - x'(t_n) \cdot h}{h} = \frac{1}{2} h x''(\xi_n) = \mathcal{O}(h^2)$$

\uparrow
 $\xi_n \in [t_n, t_{n+1}]$

$$\tau_{\text{trunc}}(h) = \tau_{n+1}$$

E.I

$$\tau_{\text{trunc}}(h) = \frac{x_{n+1} - x_n - h \phi^{\mathcal{E}_I}}{h} = \frac{x_{n+1} - x_n - h \phi_{n+1}}{h}$$

$$= \frac{x(t_{n+1}) - x(t_n) - h x'(t_{n+1})}{h} = \frac{x(t_{n+1} + (-h)) - x(t_{n+1}) - h x'(t_{n+1})}{h}$$

Taylor 2^e $x''(\xi_n) \frac{(h)^2}{2}$ $\xi_n \in [t_n, t_{n+1}]$

Taylor pour fonction $g(h) = X(t_{n+1} - h)$

$$g(h) = g(0) + g'(0) \cdot h + \frac{g''(\xi)}{2} h^2$$

$$X(t_{n+1}) = X(t_n + h) = X(t_n) + \frac{X'(t_n)}{1} h + \frac{X''(\xi)}{2} h^2$$

$$X(t_{n+1}) = X_n + h X'(t_n) + \frac{h^2}{2} X''(\xi_1)$$

$$X'(t_{n+1}) = X'(t_n) + h X''(\xi_2)$$

$$\tau_{n+1}(h) = \frac{X_{n+1} - X_n - h X'(t_n)}{h} = \frac{1}{h} (X_n + h X'(t_n) + \frac{h^2}{2} X''(\xi_1) - X_n - h X'(t_n) - h^2 X''(\xi_2)) = h \left(\frac{X''(\xi_1)}{2} - X''(\xi_2) \right) = O(h)$$

CM, U : en cours en TD

$o(1)$ ~~$O(1)$~~

Thm 2.10 $\phi_n = |W_n|$ $p_s = h |f_s|$

$k_s = h \Lambda$ $\exp(\sum k_s) = \exp(h \Lambda \cdot n) < e^{\Lambda \cdot T}$

$$g_0 + \sum p_s \leq \left(\sum_{s=0}^{n-1} h |f_s| \right) \leq \left(\sum_{s=0}^{n-1} h \cdot \varepsilon \right) \leq n \cdot h \cdot \varepsilon = T \cdot \varepsilon$$

EE vertraue hyp Thm 2.10 : f ϕ Lipschitz
 \checkmark ?





$\phi = ?$ $U_{n+1} = U_n + h \phi(t_n, U_n, f(t_n, U_n, h))$ } $\phi^{EE} = f(t_n, U_n)$
 $\phi = f$
EE $U_{n+1} = U_n + h f(t_n, U_n)$

$|\phi(t, X, f(t, X, h)) - \phi(t, Y, f(t, Y, h))| = |f(t, X) - f(t, Y)| \leq L_f |X - Y|$
~~EE~~ ou $L_f = \text{cst de Lipschitz de } f$.

EI $U_{n+1} = U_n + h \underbrace{f(t_{n+1}, U_{n+1})}_{\phi(t_n, U_n, f(t_n, U_n, h))}$

Il faut démontrer $|f(t_{n+1}, U_{n+1}) - f(t_n, U_n)| \leq C |U_n - U_{n+1}|$
 ou U_{n+1} est EI appliqué à V_n .

$V_{n+1} = V_n + h f(t_{n+1}, V_{n+1})$

 \rightarrow $\boxed{-}$ () =  + $h f(t_{n+1}, \boxed{-}$ ())

\rightarrow Heun : TD : Lipschitz

\rightarrow CM : Exo 2.6

\rightarrow ordre erreur troncature schéma de Heun (cf. 2.0)

$X_{n+1} = X_n + \frac{h}{2} \left(\underbrace{f(t_n, X_n)}_{f'_n} + \underbrace{f(t_{n+1}, X_n + h f(t_n, X_n))}_{\parallel} \right) + h z$

Par développement de Taylor :

$$f(t_{n+1}, x_{n+1}) = f(t_n, x_n) + \underbrace{d_t f(t_n, x_n)} \cdot h + \underbrace{d_x f(t_n, x_n)} \cdot h \Delta_n$$

$$h \Delta_n = \underbrace{f_n + d_t f(t_n, x_n) \cdot h + d_x f(t_n, x_n) \cdot h \Delta_n}$$

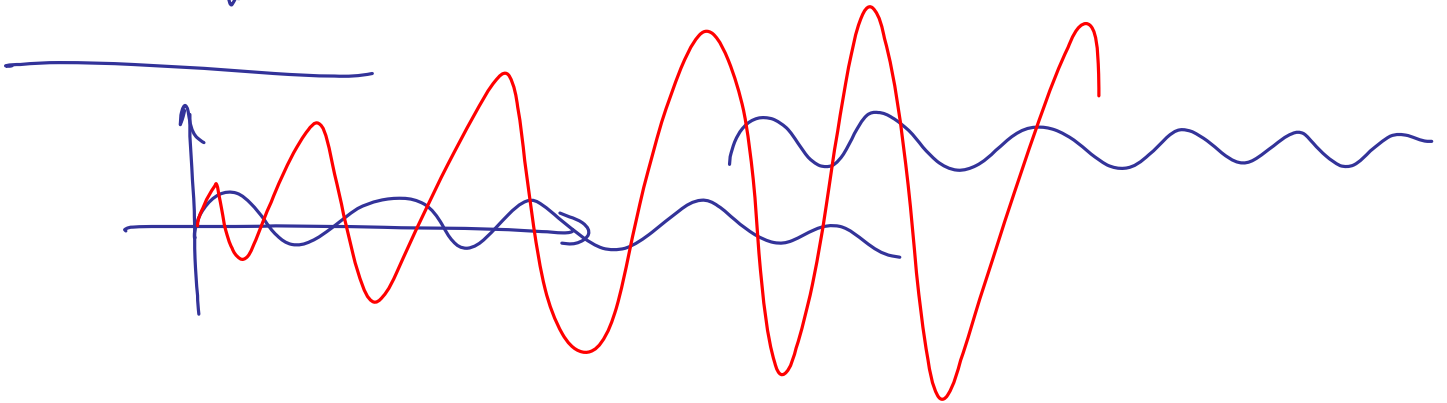
on aura donc

$$\cancel{x_n + h \cancel{x'_n} + \frac{h^2}{2} \cancel{x''_n} + \frac{h^3}{3} \cancel{x^{(3)}_n}(\xi)} = \cancel{x_n + \frac{h}{2} \cancel{f_n} + \frac{h}{2} (f_n + d_t f \cdot h + d_x f \cdot h \Delta_n)} + h \tau$$

et ainsi $\frac{h^2}{2} x''_n + O(h^3) = \frac{h^2}{2} \underbrace{d_t^2 f(t_n, x_n)} + \frac{h^2}{2} \underbrace{d_x^2 f \cdot f_n} + h \tau$

$$x''_n = x''(t_n) \equiv \frac{d^2 f(t, x(t))}{dt^2} \Big|_{t=t_n} = d_t^2 f(t_n, x_n) + d_x^2 f(t_n, x_n) \cdot \underbrace{x'_n}_{f_n}$$

Il reste que $O(h^3) = h \tau$ donc $\tau = O(h^2)$.



$$\left| \frac{1+z/c}{1-z/c} \right| < 1 \Leftrightarrow \left| 1 + \frac{(x+iy)}{z} \right|^2 < \left| 1 - \frac{(x+iy)}{z} \right|^2$$

$$\Leftrightarrow |(z+x)+iy|^2 < |(z-x)-iy|^2 \Leftrightarrow |(z+x)+iy|^2 < |(z-x)-iy|^2$$

$$\Leftrightarrow (z+x)^2 + y^2 < (z-x)^2 + y^2 \Leftrightarrow 2x < -2x \Leftrightarrow \underline{x < 0}$$

$$z = h\lambda = x+iy$$

Théorème ^{region de} Stabilité = $\{ z \in \mathbb{C}, z = h\lambda \mid U_n \rightarrow 0 \}$

$$U_{n+1} = U_n + \frac{h}{2} \{ f_n + f(t_{n+1}, U_{n+1}) \} \underline{\underline{f(t, \lambda) \rightarrow f}}$$

$$U_{n+1} = U_n + \frac{h}{2} \left[\lambda U_n + \lambda (U_n + h \lambda U_n) \right] = \left[1 + \frac{z}{2} + \frac{z}{2} (1+z) \right] U_n$$

Sans région de stabilité: $\{z \in \mathbb{C} \mid |1 + z + \frac{z^2}{2}| < 1\}$.

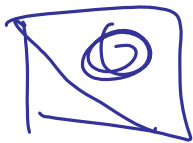
k-k $k_1 = f(t_n + c_1 h, U_n + h \sum_{j=1}^s a_{1j} k_j)$

Si $a_{1j} = 0 \forall j$ k_1 est explicite (fonction de U_n).

$k_2 = f(t_n + c_2 h, U_n + h \sum_{j=1}^s a_{2j} k_j)$ Si $a_{2j} = 0 \forall j \geq 2$

k_2 est explicite (fonction de U_n, k_1)

$\Sigma(a_{ij})$ est tel que $a_{ij} = 0 \forall j \geq i$: schéma explicite.



Dem prop 2.24 p 13

On doit montrer $\tau(h) = o(1)$
(def 2.8 p 20)

$$\tau(h) = \frac{X(t_{n+1}) - U_{n+1}}{h} = \frac{X(t_{n+1}) - [X(t_n) + h \sum_{i=1}^s b_i f(t_n + c_i h, U_n + \sum_{j=1}^s a_{ij} k_j)]}{h}$$

$$= \frac{X(t_{n+1}) - X(t_n) - h \sum_{i=1}^s b_i (f(t_n, U_n) + \theta(h))}{h} \quad \text{Taylor}$$

$$= X'(t_n) \cdot \frac{h}{h} - \frac{h \sum_{i=1}^s b_i X'(t_n)}{h} + \frac{\theta(h) \cdot h}{h}$$

$$= X'(t_n) (1 - \sum_{i=1}^s b_i) + \theta(h) \quad \text{Pour consistance il faut et}$$

\Rightarrow suffit d'avoir $\boxed{\sum_{i=1}^s b_i = 1}$

$$Y_{n+k} \sim X(t_{n+k})$$

$$Y_0 \sim X|_0$$

$t_1 \dots$

ET comme méthode multi-pos or $\ell=3$

$$U_{n+1} = U_n + h f_n \Leftrightarrow U_{n+1} - U_n = h f_n \Leftrightarrow (-1)U_n + (+1)U_{n+1} = h \cdot [1 \cdot f(t_n, U_n) + 0 \cdot f(t_{n+1}, U_{n+1})]$$

\downarrow b_0 \downarrow 1 \downarrow a_0 \downarrow a_1

Consistence Adam-Bashforth

1) $\sum_{i=0}^2 a_i = a_0 + a_1 + a_2 = 0 + (-1) + 1 = 0 \quad \checkmark$

2) 2^e condition $\sum_{k=0}^2 b_k = \sum_{k=0}^2 a_k \cdot k$, mais ($S=2$)

$$\sum_{k=0}^2 b_k = b_0 + b_1 + b_2 = -\frac{1}{2} + \frac{3}{2} + 0 = 1$$

$$\sum_{k=0}^2 k a_k = 0 a_0 + a_1 \cdot 1 + 2 a_2 = 0 + 1(-1) + 2 \cdot 1 = 1$$

} 2^e cond ok.

Remarque on peut montrer que Adam-Bashforth & BDF sont d'ordre ≥ 2 car $\tau(h) = \mathcal{O}(h^2)$

Ex BDF Il faut montrer que $\frac{X_{n+2} - \frac{1}{3}X_{n+1} + \frac{2}{3}X_n - \frac{h}{3}f(t_{n+2}, X_{n+2})}{h}$

$= \mathcal{O}(h^2)$ ou encore $\frac{1}{h} (X(t_{n+2}) - \frac{1}{3} (X(t_{n+2}) - hX'(t_{n+2})) + \frac{2}{3} X(t_{n+2}) + \frac{h^2}{2} X''(t_{n+2})) + \mathcal{O}(h^3)$

X_{n+2}

$$X(t_{n+2} - h) = X(t_{n+2}) - hX'(t_{n+2}) + \frac{h^2}{2} X''(t_{n+2}) + \mathcal{O}(h^3)$$

$$X(t_{n+2} - 2h) =$$

$$\frac{1}{3} (X(t_{n+2}) - 2hX'(t_{n+2}) + \frac{4h^2}{2} X''(t_{n+2}) - \frac{2}{3} X'(t_{n+2})) \xrightarrow{\text{à montrer}} \mathcal{O}(h^2)$$

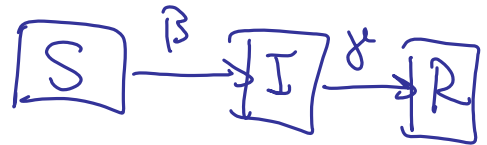
? $\mathcal{O}(h)$

$$s(t) = \frac{S(t)}{N}, \quad i(t) = \frac{I(t)}{N}, \quad r(t) = \frac{R(t)}{N}$$

$s + i + r = 1 \quad \forall t. \quad S(t) + I(t) + R(t) = N \quad \forall t > 0$

$$\frac{S(t) - S(t+\Delta t)}{\Delta t} \approx \beta S \frac{I}{N} - S'(t) = \beta S \frac{I}{N}$$

$$S'(t) = -\beta S \frac{I}{N}$$



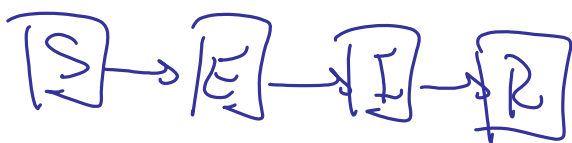
$$\frac{I(t+\Delta t) - I(t)}{\Delta t} \approx \beta S \frac{I}{N} - \gamma I \quad \left| \frac{1}{\Delta t} \right.$$

$$\begin{aligned} \dot{I} &= \beta S \frac{I}{N} - \gamma I \\ \dot{R} &= \gamma I \end{aligned}$$

$$\beta \leftrightarrow \frac{\beta}{N}$$

$$\begin{cases} \dot{s} = -\beta s i / N \\ \dot{i} = \beta s i / N - \gamma i \\ \dot{r} = \gamma i \end{cases}$$

Si on divise chaque eq par N :

$$\begin{cases} \dot{s} = -\beta s i \\ (\dot{i})' = \beta s i - \gamma i \\ \dot{r} = \gamma i \end{cases}$$


$$\underbrace{\frac{S}{N}}_{s \approx 1} > \frac{\gamma}{\beta}$$

$$R_0 = \beta / \gamma$$

$$R_0 > 1$$

$$\frac{\beta S}{N} > \gamma \quad \uparrow$$

$$\frac{S}{N} > \frac{1}{R_0}$$

$$R_0 > \frac{2[S]}{1}$$

$$I = I(\dots; S_0)$$

$$R_0 > \frac{N}{S}$$

$$\frac{\beta}{\delta} > \frac{N}{S(t)}$$

$$R_t = \frac{N}{S(t)}$$

$$\frac{\beta}{\delta} > \frac{N}{S_0} > 20$$

$$S_0 = N/2 \quad N/S = 2 \quad \boxed{S_0 = 5\%} \quad N/S = 20$$

$$\forall t: R(t) \in I(t) \in S(t)$$

$$\hookrightarrow R_\infty \in I_\infty \in S_\infty = R_0 \in I_0 \in S_0$$

$$R_1 \delta = 1st$$

$$\boxed{R_\infty = I_0 \in S_0 - S_\infty}$$

Questions math - montrer $S, I, R > 0 \forall t$

$$t = t(S)$$

$$\Leftrightarrow S$$

par le modèle SIR

$$\frac{\dot{I}}{S} = \frac{\beta S I / N - \delta I}{\beta S I / N}$$

$$\frac{dI}{dt} \frac{dS}{dt}$$

$$\boxed{I'(S) = -1 + \frac{\delta N}{\beta S}}$$

$$I(S) = I(S=S_0)$$

$$+ \int_{S_0}^S I'(u) du \Rightarrow I(S) = I_0 + \int_{S_0}^S -1 + \frac{\delta N}{\beta} \cdot \frac{1}{S} dS$$

$$I(t) = I_0 + (S_t - S_0) + \frac{\delta N}{\beta} \ln\left(\frac{S_t}{S_0}\right) \Rightarrow S = \dots$$

$$I_\infty = I_0 + S_\infty - S_0 + \frac{\delta N}{\beta} \ln\left(\frac{S_\infty}{S_0}\right) \text{ donc}$$

$$0 = \frac{N}{S_\infty} - R_\infty$$

$$\parallel \frac{N}{S_\infty} - R_\infty = N - S_\infty$$

$$0 = I_0 + N - S_\infty - S_0 + \frac{\delta N}{\beta} \ln\left(\frac{N - S_\infty}{S_0}\right)$$

$$1 - \frac{S_\infty}{S_0} = e^{-R_0(S_\infty - S_0)}$$

$$S_\infty = S_0 \quad 1 = e^{-R_0(S_\infty - S_0)} \quad \text{Now}$$

$S = S_0$ est-elle solution?

$0 = e^{-\beta_0(S_0 + I_0)}$ non

↳ immunité de groupe

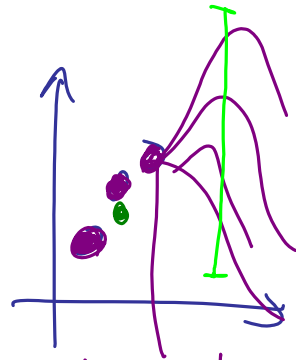
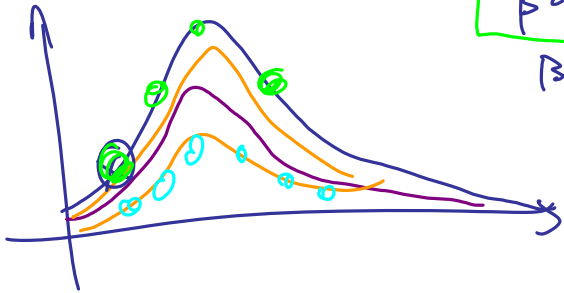
$I_0 \gg 1$
 S_0, R_0

β, γ

$\beta = 1/\tau \quad \gamma = 1/\tau_c \quad R_0 = \tau/\tau_c$

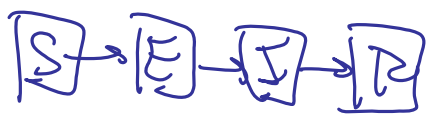
$\beta = 1/12 \quad \gamma = 1/15 \quad R_0 = 2.5$

CALIBRATION



mesures
prédiction

Questions math • calculer R_0 (formule)



$$\begin{cases} \dot{S} = -\beta SI \\ \dot{E} = \beta SI - \gamma E \\ \dot{I} = \gamma E - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

$R_0 = ?$

"next generation"

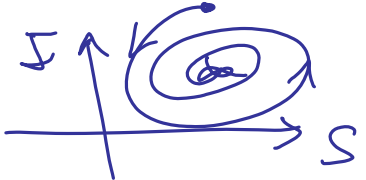
• calculer les points stables

SIR + natalité

$\rightarrow \mu$ naissances / décès

$$\begin{cases} \dot{S} = \mu - \beta SI - \mu S \\ \dot{E} = \beta SI - \gamma E - \mu E \\ \dot{I} = \gamma E - \mu I \end{cases}$$

$I_\infty \neq 0$
 $S, E, R \rightarrow S_\infty, I_\infty, R_\infty$



$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h^2)$$

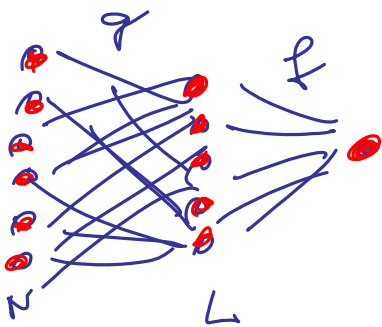
$$\frac{f(x+h) - f(x)}{h} = f'(x) + O(h^2)$$

$$\frac{f(x) + f'(x) \cdot h + f''(\xi) \frac{h^2}{2} - f(x)}{h} = f'(x) + \underbrace{\frac{f''(\xi)}{2} \cdot h}_{O(h)} \quad \checkmark$$

$$\frac{f(x+h) - f(x-h)}{2h} \rightarrow \frac{1}{2h} \left\{ \cancel{f(x)} + \cancel{f'(x)h} + \cancel{f''(x)\frac{h^2}{2}} + \frac{f^{(3)}(\xi_+)}{6} h^3 - \cancel{f(x)} + \cancel{f'(x)h} - \cancel{f''(x)\frac{h^2}{2}} + \frac{f^{(3)}(\xi_-)}{6} h^3 \right\}$$

$$= f'(x) + \underbrace{\frac{f^{(3)}(\xi_+) + f^{(3)}(\xi_-)}{6} h^2}_{O(h^2)}$$

$$\mathbb{R}^M \xrightarrow{g} \mathbb{R}^L \xrightarrow{f} \mathbb{R}$$



$$F = f \circ g$$

$$\left(\mathcal{J}_x F \right)_{t_j} = \frac{\partial F_i}{\partial x_j}$$

$$\frac{\partial U_{114}}{\partial U_{114}} = \frac{\partial f}{\partial U_{114}} = \frac{\partial f(U_{114}, U_6)}{\partial U_{114}} = -1$$

$$\frac{\partial U_6}{\partial U_6} = \frac{\partial f}{\partial U_6} = \frac{\partial f(U_{114}, U_6)}{\partial U_6} = 1$$

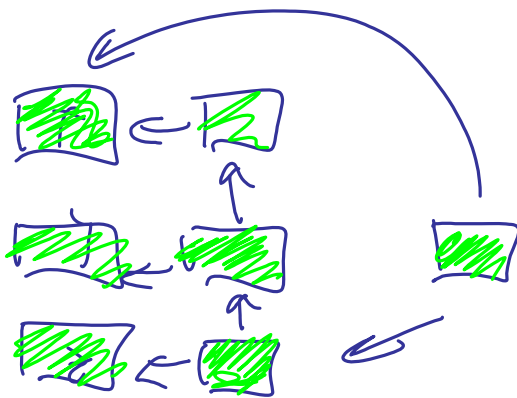
$$\delta U_{113} = \frac{\partial f}{\partial U_{113}} = \frac{\partial f}{\partial U_{113}} (U_{114} \quad U_6) =$$

$$\underbrace{\frac{\partial f}{\partial U_{114}}}_{\delta U_{114} = -1} \cdot \underbrace{\frac{\partial U_{114}}{\partial U_{113}}}_1 + \underbrace{\frac{\partial f}{\partial U_6}}_{\delta U_6 = 1} \cdot \underbrace{\frac{\partial U_6}{\partial U_{113}}}_0 = (-1) \cdot 1 + 1 \cdot 0 = -1$$

Sur le graphe initial

$$\delta U_7 = \frac{\partial f}{\partial f} = 1$$

$$\delta U_6 = \frac{\partial f}{\partial U_6} = 1$$



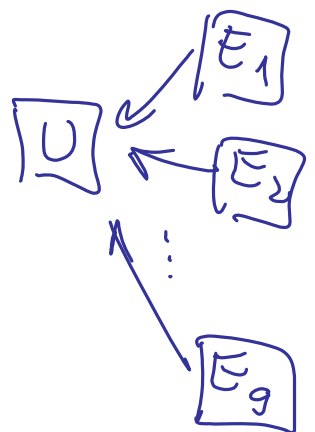
$$\begin{aligned} \delta U_5 &= \frac{\partial f}{\partial U_5} = \frac{\partial f(U_6, U_1)}{\partial U_5} \\ &= \frac{\partial f}{\partial U_6} \cdot \frac{\partial U_6}{\partial U_5} + \frac{\partial f}{\partial U_1} \cdot \frac{\partial U_1}{\partial U_5} \\ &= \delta U_6 \cdot \mu_3 \end{aligned}$$

~~$\frac{\partial f}{\partial U_1} \cdot \frac{\partial U_1}{\partial U_5}$~~

δU_1

$$\delta U = \sum_{E_k} \delta E_k \cdot \frac{\partial E_k}{\partial U}$$

noeuds dont proviennent
les flèches entrant de le
graphe backward



ou encore { tous les noeuds qui
utilisent la valeur de U
de le graphe directe }

$$\tilde{y}_2 = w_2 y_1 + b_2$$

$$\frac{\partial \tilde{y}_2}{\partial w_2} = y_1$$

$$\frac{\partial \tilde{y}_2}{\partial b_2} = \text{Id}$$

$$\tilde{y}_1 = w_1 y_0 + b_0$$

$$\frac{\partial \tilde{y}_1}{\partial w_1} = y_0$$

$$a - b = 1$$

$$\begin{array}{r} \cancel{3-2} \\ 3-2 \\ 1-0 \end{array}$$

$$\begin{array}{r} 1000001 - 1000000 \\ = 1 \end{array}$$