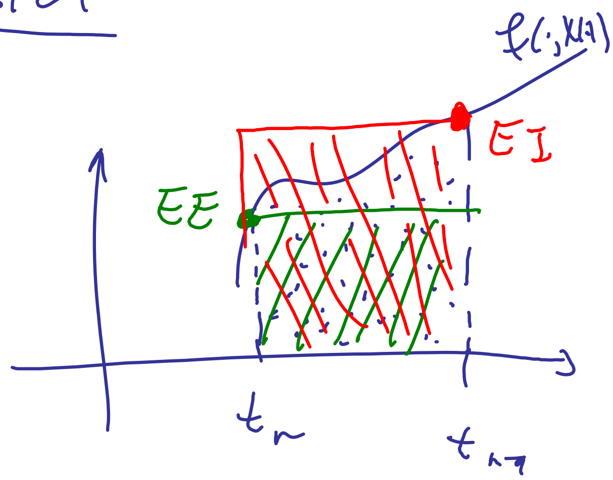
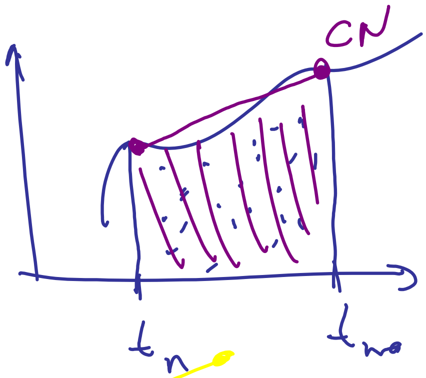
Analyse numérique, M1, 2023/24

G. TURINICI

$$X'(t) = f(t, X(t))$$

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} f(s, X(s))\,ds$$

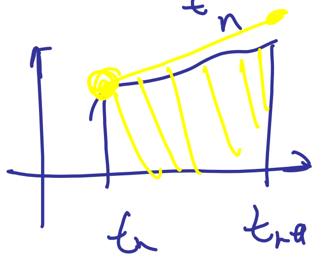




$\sim h f_n$ (EE)

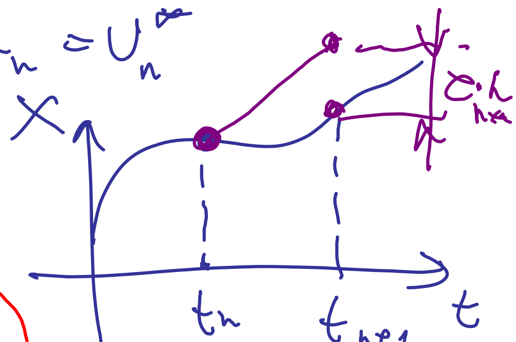
$\sim h f_{n+1}$ (EI)

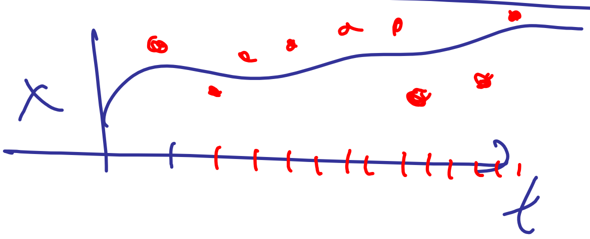
$\sim \frac{h}{2}(f_n + f_{n+1})$ (CN)



$$U^*_{n+1} = U^*_n + h\,\Phi(t_n, f_n, U^*_n, h) \quad \text{pour } X_n = U^*_n$$

$$\boxed{U^*_{n+1} = X_n + h\,\Phi(t_n, X_n, f(t_n, X_n), h)}$$





$$\boxed{\tau_{n+1} = \frac{X_{n+1} - U^*_{n+1}}{h}}$$

Erreur troncature EE

$$\Phi(\dots) = f_n = f(t_n, U_n)$$

$$X_{n+1} = X_n + h\,\underbrace{f(t_n, X_n)} + h\,\tau_{n+1}$$

$$X(t_n + h) = X(t_n) + h\cdot X'(t_n) + h\,\tau_{n+1}$$

$$t_{n+1} = t_n + h$$

$\varepsilon = 10^{-16}$ $\quad \varepsilon^{1/2}$ $\quad 10^{-8}$ $\quad (\varepsilon^{1/2})^{1/2} = 10^{-4}$

Pour schéma de Heun : $\phi = \dfrac{f_n + f(t_{n+1}, u_n + h f_n)}{2}$

On veut que ϕ soit Lipschitz c'est à dire $\exists \Lambda$ tq :

$$f(t_n, y) + f(t_{n+1}, y + h f(t_n, y)) - \left[f(t_n, \xi) + f(t_{n+1}, \xi + h f(t_n, \xi))\right] \le \Lambda |y - \xi|$$

On a pour $\Lambda = L + L(1 + hL) = 2L + hL^2$

(L = cst de Lipschitz de f).

$$\le L|y - \xi| + L|y + hf(t_n, y) - \xi - hf(t_n, \xi)|$$

$$\le L|y - \xi| + L\left[|y - \xi| + h\,|f(t_n, y) - f(t_n, \xi)|\right]$$

$$\le \Lambda |y - \xi|$$

Pour EI cf. 2.10.1 p 21/22

$$\phi(t_n, U_n, f, h) - \phi(t_n, V_n, f, h) \le \frac{L}{1 - hL} |U_n - V_n|$$

Lipschitz pour $h < 1/L$.

	Sol numérique U_n	Sol exacte $X(t)$
eq exacte $x' = f(t, x)$	on la veut $\approx$ satisfaite	exacte
eq du schéma num $U_{n+1} = U_n + h\phi$	exacte	erreur troncature ε

(2-11) est Taylor à l'ordre 1 avec reste exacte d'ordre 2 pour la fonction $g(t) = x'(t)$ autour de $t = t_n$

$$g(t_n+h) = g(t_n) + g'(t_n)\cdot h + \frac{h^2}{2} g''(\eta) \qquad \eta \in [t_n, t_{n+1}]$$

$$x'(t_{n+1}) = x'(t_n) + x''(t_n)\cdot h + \frac{h^2}{2} x^{(3)}(\eta)$$

($t_{n+1} = t_n + h$)

TD corollaire 2.13 pour Heun, ~~Ex1~~

$$X_{n+1} = X_n + h f(t_{n+1}, X_{n+1}) + h \tau_{n+1}$$

$$X_n = X_{n+1} + (-h) X'_{n+1} + \frac{(-h)^2}{2} X^{(2)}(\xi)$$

(Taylor au point $t = t_{n+1}$, incrément $-h$, fonction $g(t) = X(t)$).

$$X_{n+1} = X_n + h X'_{n+1} - \frac{h^2}{2} X^{(2)}(\xi)$$

Donc $\tau_{n+1} = \frac{-h}{2} X^{(2)}(\xi)$ donc $\tau = O(h)$ ✓

$$\boxed{g(t+(-h)) = g(t) + (-h) g'(t) + \frac{(-h)^2}{2} g^{(2)}(\xi)}$$

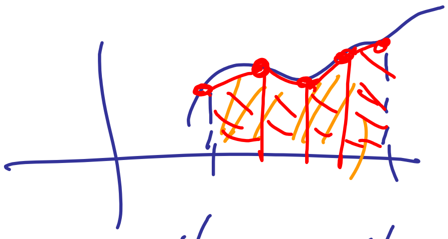
Région de stabilité du schéma de Heun Ex 2.18/25

$$U_{n+1} = U_n + \frac{h}{2}\left(f_n + f(U_n + h f_n)\right) \qquad (\text{cf p } 18)$$

$$U_{n+1} = U_n + \frac{h}{2}\left(\lambda U_n + \lambda(U_n + h\lambda U_n)\right) \overset{z = h\lambda}{=\!=} U_n + \frac{z}{2} U_n + \frac{z}{2}(U_n + zU_n) = U_n\left(1 + \frac{z}{2} + \frac{z}{2}(1+z)\right) = U_n\left(1 + z + \frac{z^2}{2}\right).$$

Région de stabilité $\{z = h\lambda \mid |1 + z + \frac{z^2}{2}| < 1\}$

Runge - Kutta



$$\begin{pmatrix} \diagdown & 0 \end{pmatrix}$$

Si A est triangulaire inf strict :

(2.18) $$k_1 = f(t_n + \underbrace{c_1}_{0=\sum a_{1j}} h, \underbrace{U_n + h \sum_{j=1}^{s} \underbrace{a_{1j} k_j}_{0}}_{U_n})$$

$$k_1 = f(t_n, U_n)$$

$$k_i = f(t_n + c_i h, U_n + h \sum_{\ell=1}^{i-1} a_{i\ell} k_\ell)$$

schéma explicite on calcule dans l'ordre $k_1, k_2, k_3, \ldots$

dep que de k_1

dep que du k_1, k_2

Si méthode semi-implicite $A_{ij} = 0$ si $j > i$

$$\begin{pmatrix} \cdot & \diagdown & 0 \\ \vdots & \cdot & \\ \cdot & \cdots & \cdot \end{pmatrix} \qquad k_i = f\left[t_n + c_i h, U_n + h\left(\sum_{\ell=1}^{i-1} a_{i\ell} k_\ell + a_{ii} k_i\right)\right]$$

1 équation en k_i à résoudre après avoir trouvé $k_1, \ldots, k_{i-1}$!

Si méthode implicite : système de s équations à résoudre.

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

EE $\quad U_{n+1} = U_n + h f_n$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1 & 0 \end{array}$$

$$\boxed{U_{n+1} = U_n + h f(t_{n+1}, U_{n+1})}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline & 0 & 1 \end{array}$$

$F = \sum b_i k_i$

$k_i = f(t_{n+1}, U_{n+1})$

$\boxed{U_n + h f_{n+1}} = f(t_{n+1}, U_{n+1})$

$c_2 = 1$

$k_2 = f(t_n + h, U_n + h \underbrace{f_{n}}_{k_1})$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$F = \underbrace{b_1}_{1} k_1$

$U_{n+1} = U_n + hF$

$U_{n+1} = U_n + h \cdot 1 \cdot k_1$

$U_{n+1} = U_n + h \underbrace{f(\ldots)}_{k_1}$

$k_1 = f(t_n + h, U_n + h k_1)$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ a & a & 0 \\ \hline & \frac{2a-1}{2a} & 1/2a \end{array}$$

$b_1 + b_2 = 1$

$a b_2 = \frac{1}{2}$ $\qquad b_2 = \frac{1}{2a}$

$b_1 = 1 - \frac{1}{2a}$

$a = 1$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 1/2 & 1/2 \end{array}$$

$U_{n+1} = U_n + hF$

$F = b_1 k_1 + b_2 k_2 \qquad k_1 = f_n$

$k_2 = f(t_{n+1}, U_n + h \underbrace{f_n}_{k_1})$

$U_{n+1} = U_n + \frac{1}{2} f_n + \frac{1}{2} f(t_{n+1}, U_n + h f_n)$ Heun!

Dem prop 2.24

$\tau = \dfrac{Y_{n+1} - U_{n+1}^*}{h}$

$$U_{n+1}^{*} = \underbrace{x_n + h\sum_{i=1}^{s} b_i K_i}$$

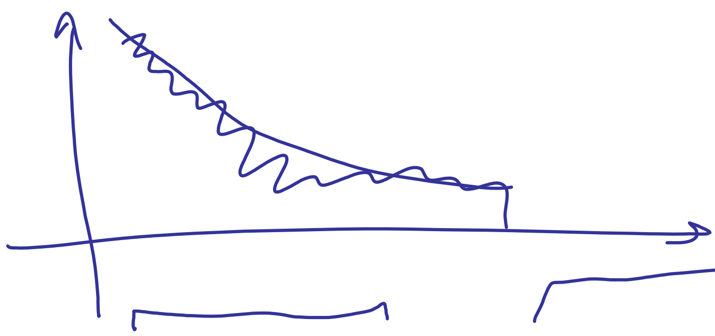
$$K_i = f\left(t_n + c_i h, x_n + h\sum_{j=1}^{s} a_{ij} K_j\right) = f(t_n, x_n) + O(h)$$

$$\tau_{n+1}(h) = \frac{x(t_n+h) - \left[x_n + h\sum\left(b_i f_n + O(h)\right)\right]}{h}$$

$$\frac{x(t_n+h) - x(t_n) - h f_n \sum b_i + O(h^2)}{h} =$$

$$= \frac{h x'(t_n) + O(h^2) - h\sum b_i x'(t_n)}{h} = \underbrace{x'_n\left(1 - \sum b_i\right)}_{\substack{\text{cv vers } 0 \\ \text{ssi } \sum_{i=1}^{s} b_i = 1}} + \underset{\substack{\downarrow \\ 0}}{O(h)}$$

On definit la consistence des methodes implicites par $\tau(h) \xrightarrow{h\to 0} 0$ et $\tau(h) = \frac{x_{n+1} - U_{n+1}^{*}}{h}$.



$\underbrace{y_0, \ldots y_s}$, y_{s+1} $\overbrace{\underbrace{\ldots, y_m}\ y_{m+1}}^{\text{d}}$

$y_{m+1} = n+s$

Ex 2.38 BDF. $\boxed{s=2}$ $a_0 = 1/3$ $a_1 = 4/3$ $a_2 = 1$

$b_0 = 0$ $b_1 = 0$ $b_2 = 2/3$

On vérifie $\sum_{n=0}^{s} a_n = 0 = 1/3 - 4/3 + 1 = 0$ ✓ 1ère cond ok

$\sum_{n=0}^{s} k a_n = 0 \cdot a_0 + a_1 + 2a_2 = -\frac{4}{3} + 2 = \frac{2}{3}$

$\sum_{n=0}^{s} b_n = 0 + 0 + 2/3$.

égalité 2e cond de consistance ok

Par 2.35 BDF est consistant.

$s(t) = \frac{S(t)}{N} = $ % de susceptibles ds la population

$i(t) = I(t)/N$

$r(t) = R(t)/N$.

Nb de passages $I \to R$ par Δt unité de tps

$\sim \gamma \underline{I(t) \cdot \Delta t}$

$I(t+\Delta t) \simeq I(t) + \frac{\beta S I}{N} \Delta t - \gamma I \Delta t$

$\frac{I(t+\Delta t) - I(t)}{\Delta t} \simeq \frac{\beta S I}{N} - \gamma I \quad (\Delta t \to 0 \ 2.40)$

$\frac{\beta S}{N} > \gamma \Leftrightarrow \boxed{\frac{S}{N} > \frac{\gamma}{\beta} = \frac{1}{R_0}}$

$S_0 = N$

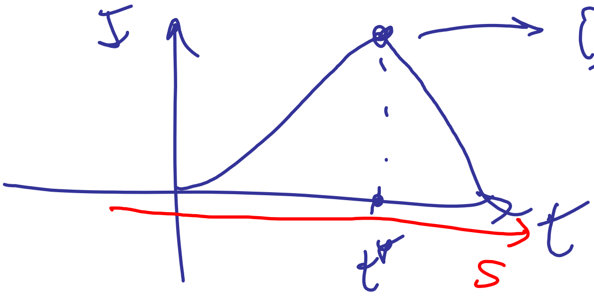
Déclenchement / début $\quad 1 > \frac{1}{R_0} \quad \boxed{R_0 > 1}$

$\| \beta / \gamma$

Si $S_0 = N\cdot(1-\lambda) : 1-\lambda > \frac{1}{R_0}$ $\qquad \boxed{1 - \frac{1}{R_0} = \lambda}$

$\underline{R_0 = 20} : 1 - \frac{1}{20} = \underbrace{95\%}$

$\underline{R_0 = 40} : 1 - \frac{1}{40} = 97{,}5\%$



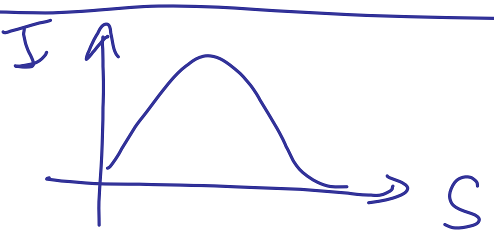
$\frac{\beta S(t^*)}{N} = \gamma$

$$\begin{cases} I' = \left(\frac{\beta S}{N} - \gamma\right) I \\ S' = -\beta \frac{SI}{N} \end{cases} \qquad \frac{I'}{S'} = \frac{\frac{dI}{\cancel{dt}}}{\frac{dS}{\cancel{dt}}} = \frac{dI}{dS}$$

$$I'(S) = \frac{+\beta SI/N - \gamma I}{-\beta SI/N} = -1 + \frac{\gamma N}{\beta S}$$

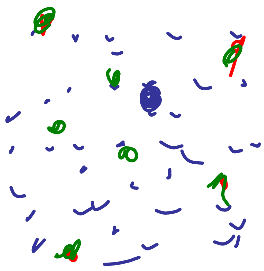
$$I(S) = I(S(0)) + \int_{S_0}^{S} \left(-1 + \frac{\gamma N}{\beta}\frac{1}{S}\right) dS = I_0 + S_0 - S + \frac{\gamma N}{\beta} \ln(S/S_0)$$

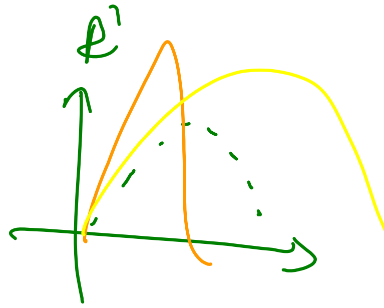
$$\boxed{I(t) = I_0 + S_0 - S(t) + \frac{\gamma N}{\beta} \ln(S(t)/S_0)}$$

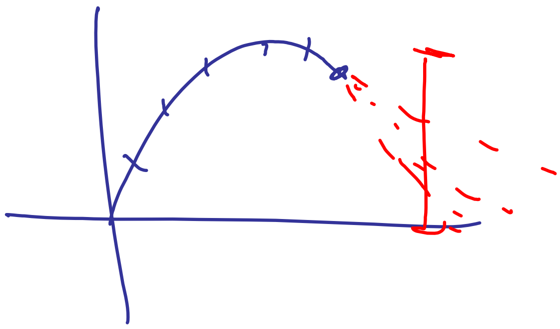


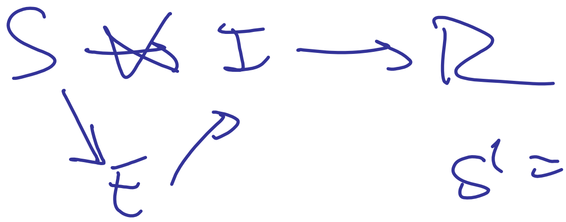
$S_\infty = \lim_{t\to\infty} S(t)$

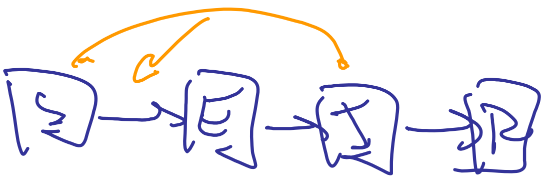
$\underbrace{S_0 - S_\infty}_{S}$ = taille de l'épidémie











$$\left.\begin{aligned} S' &= -\beta S I \\ E' &= \beta S I - \gamma_E E \\ I' &= \gamma_E E - \gamma_I I \\ R' &= \underbrace{\gamma_I I}_{=0} \end{aligned}\right\} = 0$$