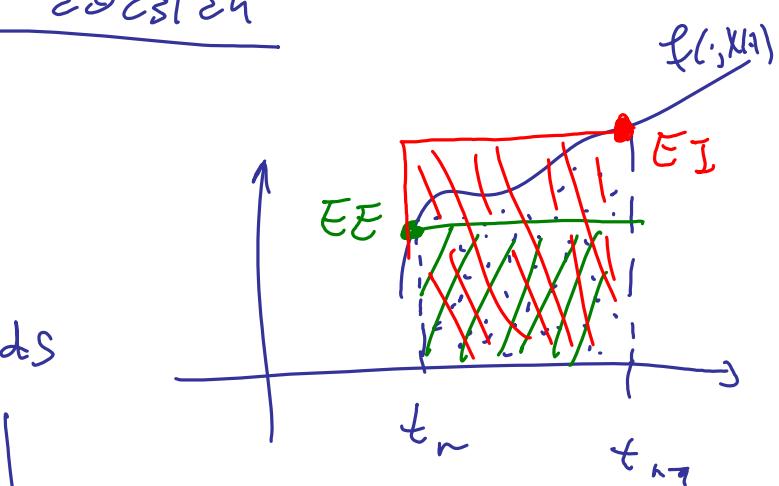
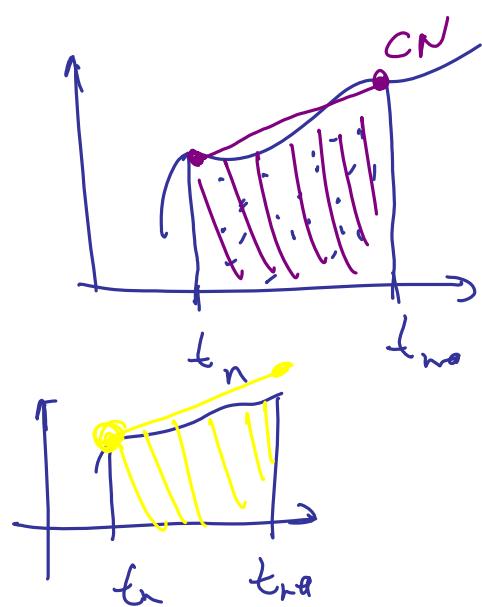


Analyse numérique , n°1, 2023/24

G.TURINICI

$$x'(t) = f(t, x(t))$$

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} f(s, x(s)) ds$$



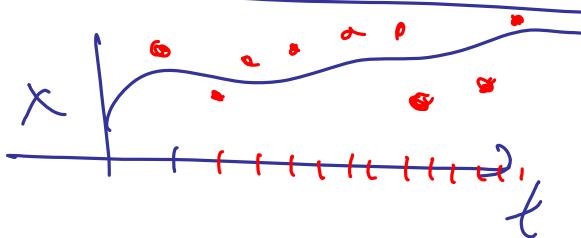
$$\approx h \ln (EE)$$

$$\approx h f_{n+1} (EI)$$

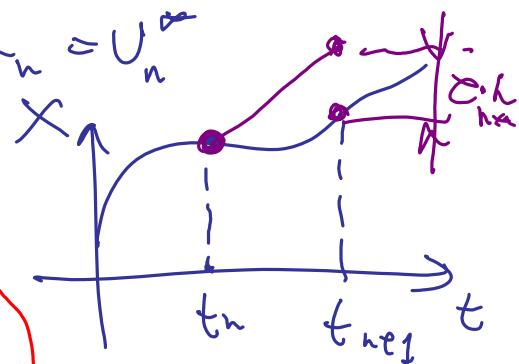
$$\approx \frac{h}{2} (f_n + f_{n+1}) (CN)$$

$$U_{n+1}^* = U_n^* + h \phi(t_n, f_n, U_n^*) \quad \text{pour } x_n = U_n^*$$

$$U_{n+1}^* = x_n + h \phi(t_n, x_n, f(t_n, x_n), h)$$



$$\sum_{n+1} = \frac{x_{n+1} - U_n^*}{h}$$



Erreur truncature EE

$$\phi(\dots) = f_n = f(t_n, U_n)$$

$$x_{n+1} = x_n + h \underbrace{f(t_n, x_n)}_{f_n} + h \sum_{n+1}$$

$$x(t_{n+1}) = x(t_n) + h \cdot x'(t_n) + h \sum_{n+1}$$

$$t_{n+1} = t_n + h$$

$$\varepsilon > 10^{-16}$$

$$\varepsilon^{1/2}$$

$$10^{-8}$$

$$(\varepsilon^{1/2})^{1/2} = 10^{-4}$$

Pour schéma de Heun : $\phi = \frac{f_n + f(t_{n+1}, u_n + hf_n)}{2}$

On veut que ϕ soit Lipschitz c'est à dire $\exists \lambda, t_2$:

$$|f(t_n, \tilde{u}) - f(t_n, u) + h f(t_n, \tilde{u})| - \\ [f(t_n, \tilde{u}) + f(t_{n+1}, \tilde{u} + h f(t_n, \tilde{u}))] \leq \lambda |u - \tilde{u}|$$

On pose $\Lambda = L + L(1 + hL) = 2L + hL^2$
 $(L = \text{cst de Lipschitz de } f)$.

$$\leq L|u - \tilde{u}| + L|u + h f(t_n, u) - \tilde{u} - h f(t_n, \tilde{u})| \\ \leq L|u - \tilde{u}| + L[(|u - \tilde{u}| + h |f(t_n, u) - f(t_n, \tilde{u})|)] \\ \leq \Lambda |u - \tilde{u}|$$

Pour EJ cf. 2.10.1 p 21/22

$$|\phi(t_n, u_n, f, h) - \phi(t_n, v_n, f, h)| \leq \frac{\Lambda}{1-hL} |u_n - v_n|$$

Lipschitz pour $h < 1/L$.

Sol numerique u_n	Sol exacte $x(t)$
exacte $u_n = f(t_n)$	exacte
exacte du schéma num $u_n = u_{n-1} + \phi$	erreur troncature ϵ

(2.11) est Taylor à l'ordre 1 avec reste exacte d'ordre 2

pour la fonction $g(t) = X(t)$ autour de $t=t_n$

$$g(t_{n+1}) = g(t_n) + g'(t_n) \cdot h + \frac{h^2}{2} g''(\eta) \quad \eta \in [t_n, t_{n+1}]$$
$$X(t_{n+1}) = X(t_n) + X'(t_n) \cdot h + \frac{h^2}{2} X''(\zeta) \quad \zeta \in [t_n, t_{n+1}]$$

TD corollaire 2.13 pour Heun, ~~Heun~~

$$X_{na} = X_n + h f(t_{n+1}, X_{na}) + h \tau_{na}$$

$\tau_n = \tau_{na} + (-h) X'_{na} + \sum_{k=2}^{n-1} X^{(k)}(\zeta)$ (Taylor au point $t=t_{na}$, incrément -h, fonction $g(t)=X(t)$).

$$X_{na} = X_n + h X'_{na} - \sum_{k=2}^{n-1} X^{(k)}(\zeta)$$

$$\text{D'où } \tau_{na} = \sum_{k=2}^{n-1} X^{(k)}(\zeta) \text{ donc } \tau = O(h)$$

$$g(t+(-h)) = g(t) + (-h) g'(t) + \sum_{k=2}^{n-1} g^{(k)}(\zeta)$$

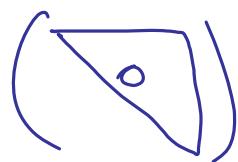
Réseau de stabilité du schéma de Heun Ex 2.18 / 25

$$U_{na} = U_n + \frac{h}{2} (f_n + f(U_n + h f_n)) \quad (\text{cf p 18})$$

$$U_{na} = U_n + \frac{h}{2} (\lambda U_n + \lambda (U_n + h \lambda U_n)) \stackrel{\lambda = h}{=} U_n + \frac{\lambda}{2} U_n + \frac{\lambda}{2} (U_n + \lambda U_n) = U_n (1 + \frac{\lambda}{2} + \frac{\lambda}{2} (1 + \lambda)) = U_n (1 + \lambda + \frac{\lambda^2}{2})$$

Réseau de stabilité $\{z \in \mathbb{C} \mid |1 + \lambda + \frac{\lambda^2}{2}| < 1\}$

Rung - Kutta



Si A est triangulaire et strict :

$$(2.18) \quad k_1 = f(t_n + c_1 h, u_n + h \sum_{j=1}^s a_{1,j} k_j)$$
$$0 = \sum_{i,j} a_{ij} \underbrace{u_n}_0,$$
$$k_1 = f(t_n; u_n)$$

$$k_i = f(t_n + c_i h, u_n + h \sum_{e=1}^{i-1} a_{i,e} k_e)$$

schéma explicatif on calcule dans l'ordre k_1, k_2, k_3, \dots

depuis que de k_1
depuis que de k_1, k_2

Si méthode semi-explicatif $A_{ij} = 0$ si $j > i$

$$\left(\begin{array}{ccccc} \vdots & \vdots & \ddots & & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \end{array} \right) \quad k_i = f(t_n + c_i h, u_n + h \sum_{e=1}^{i-1} a_{i,e} k_e + a_{ii} k_i)$$

1 équation en k_i à résoudre après avoir trouvée k_1, \dots, k_{i-1} !

Si méthode implicite : système de s équations à résoudre.



EE

$$u_{n+1} = u_n + h f_n$$

$$\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{matrix}$$

$$U_{na} = U_n + h f(t_{na}, U_{na})$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline & 0 & 1 \end{array}$$

$$F = \sum k_i K_i$$

$$K_2 = f(t_{na}, U_{na})$$

$$c_2 = 1$$

$$K_2 = f(t_{n+1}), U_{n+1} = f_{\text{new}}(U_n)$$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$F = \frac{b_1}{2} K_1$$

$$U_{n+1} = U_n + h F$$

$$\begin{array}{c} K_1 \\ \parallel \end{array}$$

$$f(t_{n+1}), U_{n+1} K_1$$

$$U_{na} = U_n + h \cdot 1 \cdot K_1$$

$$U_{new} = U_n + h f(\dots)$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \hline a & a & 0 \\ \hline \Sigma a & 1/a & \end{array}$$

$$a = 1$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline 0 & 1/2 & 1/2 \end{array}$$

$$b_1, b_2 \geq 1$$

$$a b_2 = 1 \quad \Rightarrow \quad b_2 = \frac{1}{a}$$

$$b_1 = 1 - \frac{1}{a}$$

$$U_{na} = U_n + h F$$

$$F = b_1 K_1 + b_2 K_2 \quad K_2 = f_{\text{new}}$$

$$K_2 = f(t_{n+1}, U_{n+1} \underbrace{f_{\text{new}}}_{K_2})$$

$$U_{na} = U_n + \frac{1}{2} f_n + \frac{1}{2} f(t_{na}, U_{na} + f_n) \quad \text{Blau!}$$

Denn prep 2-24

$$\tau = \frac{U_{na} - U_{na}^*}{h}$$

$$U_{n+1}^* = X_n + h \sum_{i=1}^s b_i K_i$$

$$K_i = f(t_n + c_i h, X_n + h \sum_{j=1}^{i-1} a_{ij} K_j) = f(t_n, X_n + O(h))$$

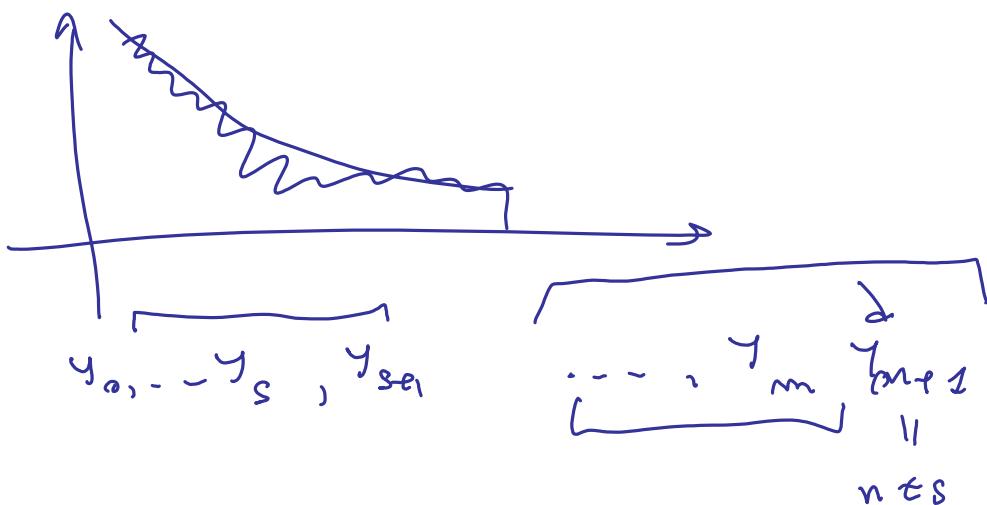
$$\epsilon_{\text{na}}(h) = \frac{X(t_n + h) - [X_n + h \sum (b_i f_n + O(h))]}{h}$$

$$\frac{X(t_n + h) - X(t_n) - h f_n \sum b_i + O(h^2)}{h} =$$

$$= h X'(t_n) + O(h^\varepsilon) - h \sum b_i X'(t_n) = \underbrace{X'_n (1 - \sum b_i)}_{\text{CV vers } 0} + O(h)$$

↓
0
ssi $\sum_{i=1}^s b_i = 1$

On définit la convergence des méthodes implicites par $\epsilon(h) \xrightarrow{h \downarrow 0} 0$ et $\epsilon(h) = \frac{X_n - U_n}{h}$.



Ex 2.33

BDF.

$$a_0 = 1/3 \quad a_1 \approx 1/3 \quad a_2 = 1$$

s=2

$$b_0 = 0 \quad b_1 = 0 \quad b_2 \approx 2/3$$

On vérifie $\sum_{k=0}^5 a_k = 1\frac{1}{3} - \frac{4}{3} + 1 = 0$ ✓

$$\sum_{k=0}^5 b_k = 0 \cdot a_0 + a_1 + 2a_2 = -\frac{4}{3} + 2 = \frac{2}{3}$$

$$\sum_{k=0}^5 b_k = 0 + 0 + \frac{2}{3}$$

1ere cond ok
égalité
2e
cond
de cohérence
ok

Par 2-3) BDF est consistant.

$$S(t) \equiv \frac{S(t)}{N} \equiv \% \text{ de susceptibles dans la population}$$

$$I(t) = I(t)/N$$

$$R(t) = R(t)/N.$$

Nb de passages $I \rightarrow R$ par Δt unité de temps

$$\sim \gamma I(t) \cdot \Delta t$$

$$I(t+\Delta t) \approx I(t) + \frac{\beta S I}{N} \Delta t - \gamma I \Delta t$$

$$\frac{I(t+\Delta t) - I(t)}{\Delta t} \approx \frac{\beta S I}{N} - \gamma I \quad (\Delta t \approx 0.24)$$

$$\frac{\beta S}{N} > \gamma \Leftrightarrow \frac{S}{N} > \frac{\gamma}{\beta} = \frac{1}{R_0}$$

$$S_0 = N$$

Déclenchement / début

$$1 > \frac{1}{R_0}$$

$R_0 > 1$

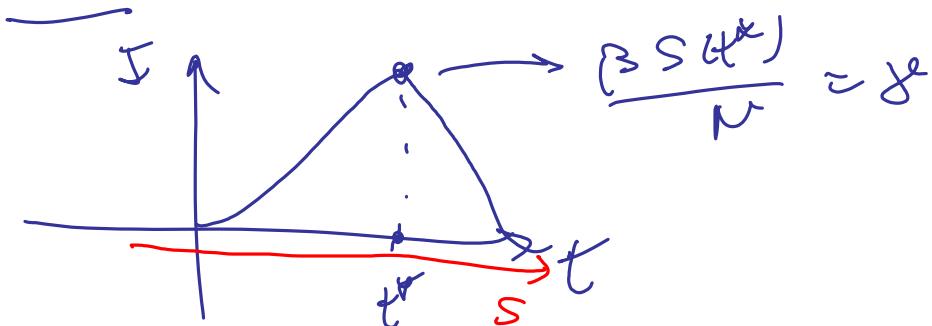
$\parallel \beta/\gamma$

$$\text{Si } S_0 = \nu \cdot (1-\lambda) : 1-\lambda > \frac{1}{S_0}$$

$$1 - \frac{1}{S_0} = \lambda$$

$$\frac{S_0 = 20}{S_0 = 20} : 1 - \frac{1}{20} = 95\%$$

$$\underline{S_0 = 60} : 1 - \frac{1}{60} = 97.5\%$$



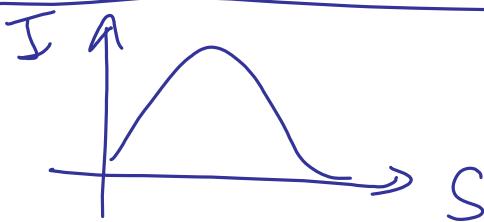
$$\begin{cases} I' = \left(\frac{\beta S I}{N} - \gamma \right) I \\ S' = -\frac{\beta S I}{N} \end{cases} \quad \frac{I'}{S'} = \frac{dI}{dt} \frac{ds}{dt} = \frac{dI}{ds}$$

$$I'(S) = \frac{+\frac{\beta S I}{N} - \gamma I}{-\frac{\beta S I}{N}} = -1 + \frac{\gamma N}{\beta S}$$

$$I(S) = I(S_0) + \int_{S_0}^S \left(-1 + \frac{\gamma N}{\beta S} \right) dS = I_0 + S_0 - S$$

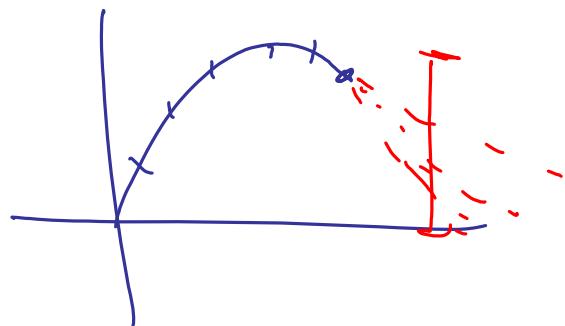
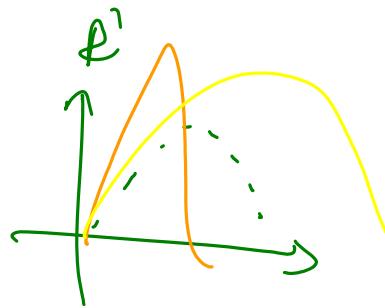
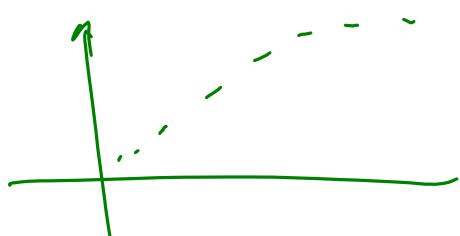
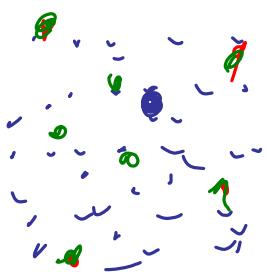
$$+ \frac{\gamma N}{\beta} \ln(S/S_0)$$

$$I(t) = I_0 + S_0 - S(t) - \frac{\gamma N}{\beta} \ln(S(t)/S_0)$$



$$S_\infty = \lim_{t \rightarrow \infty} S(t)$$

$$\frac{S_0 - S_\infty}{S_0} \approx \text{taux de l'épidémie}$$



$$S \xrightarrow{\beta SI} R$$

$$\downarrow E$$

$$S' = -\beta SI$$

$$E' = \beta SI - \gamma_E E$$

$$I' = \gamma_E E - \gamma_I I$$

$$R' = \gamma_I I$$

\Rightarrow

$= 0$

$\triangleright f = \text{ensemble de droites d'ordre } 1$

$\triangleright f^2 = 2$

$$\beta \in [0, T] \longrightarrow \mathbb{R}$$

$$\underline{\epsilon_X} \quad C(\beta) = \frac{1}{\beta} \int_{\beta(4)}^T \beta^2(t) dt \geq ?$$

$$\frac{1}{2} \left\{ \delta_{\beta^2} \right\} \Rightarrow \beta$$

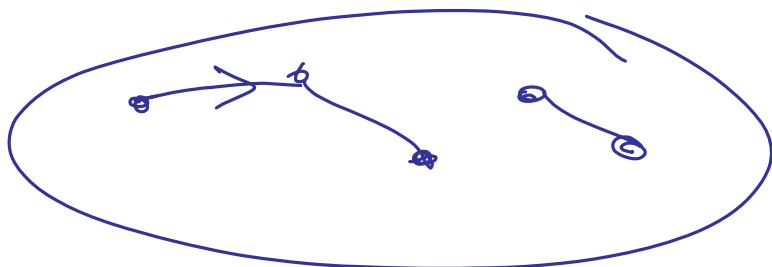
$$C(\beta + \delta_{\beta}) - C(\beta) = \underbrace{\delta_{\beta} \cdot \beta}_{\langle \delta_{\beta}, \beta \rangle} + \underbrace{\delta_{\beta}^2}_{\langle \delta_{\beta}, \delta_{\beta} \rangle}$$

$$F(x + \delta_x) = F(x) + \langle DF, \delta_x \rangle + o(\|\delta_x\|)$$

Dern formula (3,2) par Taylor

$$f(x+h) = f(x) + f'(x) \cdot h + \frac{f''(x)h^2}{2} + O(h^3)$$

$$\text{Donc } \frac{f(x+h) - f(x-h)}{2h} = \frac{\cancel{f(x)} + h f' + h^2 \cancel{\frac{f''}{2}} + O(h^3)}{\cancel{[f(x) - h f' + h^2 \cancel{\frac{f''}{2}} + O(h^3)]}} \\ = \frac{h f' + O(h^3)}{2h} = f'(x) + O(h^2).$$



$$\frac{\nabla F}{\nabla g_k} = \sum_e \frac{\nabla f}{\nabla g_e} \frac{\nabla g}{\nabla g_k} = \sum_e (\nabla f)_e \cdot (\nabla g)_{e_k} \\ = (\nabla f \cdot \nabla g)_k$$

$\boxed{\nabla F = \nabla f \cdot \nabla g}$

$$\delta \mu_6 = ? = \frac{\nabla F}{\nabla \mu_6} f = \frac{\nabla}{\nabla \mu_6} (\mu_6 - \mu_1) = 1.$$

$$\frac{\partial}{\partial g_k} (g_1 - g_2) = 1, \quad \frac{\partial}{\partial g_k} (x \cdot y) \approx 1$$

$$(U_{1,3}; U_5; U_{3,3}) \rightarrow (U_{1,4}, U_6) \rightarrow f(U_{1,4}, U_6)$$

$$\delta U_{1,3} = \frac{\partial f}{\partial U_{1,3}} = \underbrace{\frac{\partial f}{\partial U_{1,4}}}_{\text{1}} \cdot \frac{\delta U_{1,4}}{\delta U_{1,3}} + \underbrace{\frac{\partial f}{\partial U_6}}_{\text{0}} \cdot \frac{\delta U_6}{\delta U_{1,3}}$$

$$\begin{cases} g_1 = U_{1,4} \text{ comme fonction de } U_{1,3}, U_5, U_{3,3} \\ g_2 = U_6 \end{cases} \quad \dots \dots \dots$$

$$\text{Donc } \delta U_{1,3} = \delta U_{1,4} \cdot \underbrace{\frac{\delta U_{1,4}}{\delta U_{1,3}}}_{\text{1}} + \delta U_6 \cdot \underbrace{\frac{\delta U_6}{\delta U_{1,3}}}_{\text{0}}$$

$$\delta U_{1,3} = \delta U_{1,4}$$

$$\delta U_5 = \delta U_{1,4} \cdot \underbrace{\frac{\delta U_{1,4}}{\delta U_5}}_{\text{0}} + \delta U_6 \cdot \underbrace{\frac{\delta U_6}{\delta U_5}}_{U_{3,3}} = \underbrace{\delta U_6}_{\text{1}} \cdot U_{3,3} = U_{3,3}$$

$$\delta U_{3,3} = \delta U_{1,4} \underbrace{\frac{\delta U_{1,4}}{\delta U_{3,3}}}_{\text{0}} + \delta U_6 \cdot \underbrace{\frac{\delta U_6}{\delta U_{3,3}}}_{U_5} = U_5.$$

Pour les $\delta U_{1,2}, \delta U_4, \delta U_{2,2}, \delta U_{3,2}$ on utilise pareil la dérivation composée pour les fonctions

$$(U_{12}, U_4, 1U_{23}, U_{32}) \xrightarrow{g} (U_{13}, U_5, U_{313}) \xrightarrow{f} U_7 = f$$

$$\left\{ \begin{array}{l} \delta U_{12} = \delta U_{13} \cdot \underbrace{\frac{\delta U_{13}}{\delta U_{13}}}_1 + 0 = \delta U_{13} \\ \delta U_n = \delta U_5 \cdot \underbrace{\frac{\delta U_5}{\delta U_n}}_1 = \delta U_5 \\ \delta U_{23} = \delta U_5 \cdot \underbrace{\frac{\delta U_1}{\delta U_{23}}}_1 = \delta U_5 \\ \delta U_{32} = \delta U_{33} \cdot 1 = \delta U_{33} \end{array} \right.$$

Par ailleur $\delta x = \delta U_1 = \delta U_{12} - \underbrace{\delta U_{12} \cdot \frac{\delta U_{12}}{\delta U_1}}_1 + \delta U_n \cdot \underbrace{\frac{\delta U_n}{\delta U_1}}_1 = \delta U_{12} + \delta U_n \cdot \delta U_1$

$$\delta y = \delta U_{23} ; \quad \delta z = \delta U_{33}$$

Règle de calcul des dérivées : δU_a est calculable si δU_b sont déjà calculés pour tout U_b tel que $U_a \rightarrow U_b$ existe dans le graphe computationnel

$$\delta U_a = \sum_{\{b : (a, b) \in E\}} \delta U_b \cdot \frac{\delta U_b}{\delta U_a}$$

$E = A$: ensemble des arcs du graphe cf p.68

Rg 3.12

Exemple (exo 3.7) classification avec

"softmax & cross-entropy"

$$S(y) = \left(\frac{e^{y_a}}{\sum_b e^{y_b}} \right)_{a=1}^n \quad \text{fonction softmax}$$

Ex $y = \underbrace{y_0}_{\in \mathbb{R}^N}$ slot $\left(\frac{1}{1+n}, \dots, \frac{1}{n+1} \right)$ loi uniforme

$$y = (0, -\infty, \dots, -\infty) \quad S(y) = \left(\frac{1}{1+\theta}, \frac{0}{\theta+1}, \dots, \frac{0}{\theta+1} \right) \quad \text{loi de Dirac.}$$

Sortie du réseau: si on a 3 catégories "chat", "chien", "souris" la sortie sera sous forme de

loi de proba de type (p_1, p_2, p_3)

p_1 = proba que l'input soit un "chat".
ooo

Rg $\sum_{b=1}^n S(y)_b = 1 !$

Déction sera l'argmax de $S(y)$, celui qui a la proba la plus grande).

Rg Etiquettes sont de type $l = \begin{cases} \text{"chat"} \\ \text{"chien"} \\ \text{"souris"} \end{cases}$

$$\text{"chat"} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{"chien"} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{souris} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

ONE HOT ENCODING. $l_1, l_2, l_3 \in \{0, 1\}^n$

Loss "categorical cross-entropy loss"

$$(S(l) - l)^2$$

$$\text{Loss} = - \sum_{i=1}^n l_i \log(S_i) \quad \text{si } l = (l_1, \dots, l_n)^T$$

pour uncertainty: l_w

Etape 1 calcul de $S(y)$ pour cette fonction Loss.

Prop On doit minimiser $L(y, w)$

$$y_0 \rightarrow \tilde{y}_1 = w_1 y_0 + b_1 \xrightarrow{\text{A3}} \tilde{y}_2 \xrightarrow{\text{A4}} \tilde{y}_3 \rightarrow L(\tilde{y}_3, w)$$

$\underbrace{\quad}_{\text{a. min}}$

$$\delta L = 1$$

$$\delta \tilde{y}_3 = ?$$

$$Y \rightarrow - \sum_{a=1}^n l_a S(Y)_a$$

l_a est du type "one hot encoding" supposons il est non nul que pour un certain \bar{a} où il est = 1.

$$\boxed{l_a = \delta_{a\bar{a}}}$$

$$e = (e_1, \dots, e_n)^T.$$

$$\frac{\partial S(Y)_a}{\partial Y_b} = \frac{\partial}{\partial Y_b} \left(\frac{e^{Y_a}}{\sum_c e^{Y_c}} \right)$$

$$\text{Si } a \neq b \quad \frac{\partial}{\partial Y_b} \left(\frac{e^{Y_a}}{\sum_c e^{Y_c}} \right) = \frac{e^{Y_a} (\sum_c e^{Y_c}) - e^{Y_a} \cdot e^{Y_a}}{(\sum_c e^{Y_c})^2}$$

$$= S(Y)_a - S(Y)_a^2 = S(Y)_a (1 - S(Y)_a).$$

$$\text{Si } a = b : \frac{\partial}{\partial Y_b} \left(\frac{e^{Y_a}}{\sum_c e^{Y_c}} \right) = \frac{0 - e^{Y_a} \cdot e^{Y_b}}{(\sum_c e^{Y_c})^2} = - S(Y)_a S(Y)_b$$

$$\text{Donc } \boxed{\frac{\partial}{\partial Y_b} S(Y)_a = S(Y)_a (\mathbb{1}_{a=b} - S(Y)_b)}$$

$$\frac{\partial}{\partial Y_b} L(Y, w) = \frac{\partial}{\partial Y_b} \left\{ - \sum_a [l_a \log(S(Y)_a)] \right\}$$

$$= - \sum_a l_a \frac{\partial}{\partial Y_b} \log(S(Y)_a) = - \sum_a l_a \cdot (\mathbb{1}_{a=b} - S(Y)_b)$$

$$\underline{\underline{\sum l_a = 1}} \quad - l_b + S(Y)_b \cdot \text{ Donc } \boxed{\nabla L = S(Y) \cdot e}.$$

$$\boxed{\delta \tilde{y}_3 = S(\tilde{y}_3) - l_w.}$$

efficace (one hot encoded)
 $y_3 \approx S(\tilde{y}_3)$.

$$y_0 \rightarrow \tilde{y}_1 \rightarrow y_1 \quad y_1 = \text{ReLU}(\tilde{y}_1) \text{ avec } \text{ReLU}(x) = x_+.$$

$$\tilde{y}_2 = \text{ReLU}(\tilde{y}_2)$$

$$\begin{cases} \text{ReLU}(x) = x & x > 0 \\ \frac{\sqrt{x^2 + a^2} - x}{2} & x \leq 0 \end{cases}$$

$$\mathbb{R}^n \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$b_2 w_2^T = \tilde{y}_3$$

$$\delta w_2 = \delta \tilde{y}_3 \cdot \frac{\partial \tilde{y}_3}{\partial w_2} \quad (\text{cf } \star \star \star) \quad w_2: 7 \times 3$$

$$= \delta \tilde{y}_3 \cdot y_2^T \quad (\text{cf } p. 56)$$

$$\delta y_2 = \delta \tilde{y}_3 \cdot \frac{\partial \tilde{y}_3}{\partial y_2} = w_2 \cdot \delta \tilde{y}_3$$

(≥ 0) (> 0) (> 0) $\Rightarrow y_2 \neq 0$

$$\delta b_2 = \delta \tilde{y}_3 \cdot \frac{\partial \tilde{y}_3}{\partial b_2} = \delta \tilde{y}_3.$$

$$\delta \tilde{y}_2 = \delta y_2 \cdot \frac{\partial y_2}{\partial \tilde{y}_2} = \delta y_2 \odot \mathbb{1}_{\tilde{y}_2 > 0}$$

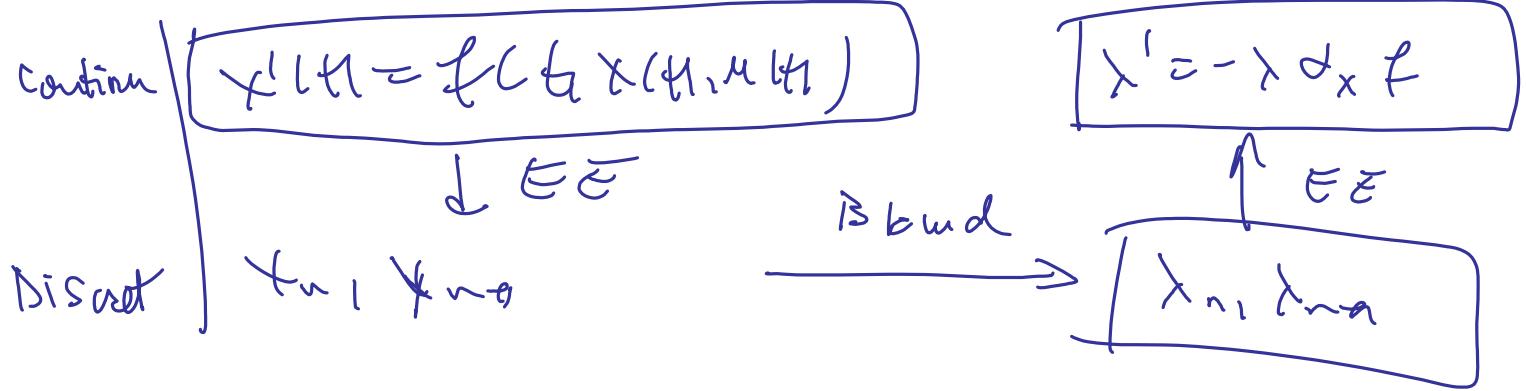
product de Hadamard i.e.
terme par terme Python &
pas " $@$ ".

Pour la suite il suffit de faire pareil comme ds le
passage $\tilde{y}_3 \rightarrow \dots \rightarrow \tilde{y}_n$.

$$\delta x_{n+1} =: x_{n+1}$$

F = fonction explicite de x_n !

$$\text{Ex: S (O1-SCT)} \quad \|x(t) - \bar{x}\|^2$$



$$(3.11) \quad x_n = x_{n-1} + h \lambda_{n-1} + \partial_x f(t_n, u_n, x_n)$$

$$(3.12) \quad x'(t) = -\partial_x f(t, x(t), u(t)) \cdot \lambda(t)$$

$$\xi(\epsilon) = \lambda(T-t)$$

$$\overline{\xi'(t)} = -\lambda'(T-t) = (-1)^{k-1} \partial_x f(T-t, x(T-t), u(T-t))$$

$$\lambda(T-t) = \overline{\partial_x f(\dots) \xi(t)}$$

$$\underline{\text{EE}} \quad \underline{x_{k+1} = x_k + h f_k} \Rightarrow \boxed{\underline{x_{k+1} = x_k + h (\partial_x f_k) \cdot \xi_k}}$$

$$T-t = n \cdot \Delta t$$

$$t = T - n \Delta t = (k-\epsilon) \Delta t$$

$$\boxed{n+k-1 = N}$$

$$\begin{aligned} \xi_{n-1} &\sim \lambda_n \\ \xi_n &\sim \lambda_{n+1} \end{aligned}$$

$$x_n = x_{n-1} + h \partial_x f \cdot \lambda_n$$

$x(T) \dots$

$$f_n = + \frac{\partial f}{\partial u} \cdot x$$

Formulation Euler-Lagrange

$$\frac{\partial}{\partial x(t)} G = \frac{\partial}{\partial x(t)} \left\{ F(x(t)) - \lambda(t) x(t) \right\}$$

$$= \mathcal{L}_x F(x(\tau)) - \lambda(\tau).$$

Colle-ci doit être nulle donc $\boxed{\lambda(\tau) = \mathcal{L}_x F(x(\tau))}$

$$x'(t) = \mathcal{L}(t)x(t), \quad x = (S, I, R).$$

$$(SIR) \quad S'(t) = -\beta \frac{SI}{N_p}, \quad I'(t) = \beta \frac{SI(t)}{N_p} - \kappa I, \quad R'(t) = \kappa I(t).$$

$$\mathcal{J}(\beta) = S(0) - S(T) + \int_0^T c(\beta(t)) dt \quad \text{à minimiser}$$

sous la contrainte (SIR) .

$$\text{et } S(t) + I(t) + R(t) = N_p \approx S(0) + I(0) + R(0) \quad (\text{population totale})$$

Donc finalement on prend $x = (S, I)$

$$\frac{\partial}{\partial \beta(t)} \int_0^T c(\beta(t)) dt = c'(\beta(t)).$$

$$F(x(\gamma)) = S(0) - S(T). \quad \text{A calculer } \frac{\partial}{\partial \beta(t)} S(T).$$

$$\text{ensuite on aura } \frac{\partial}{\partial \beta(t)} \mathcal{J}(\beta) = - \frac{\partial}{\partial \beta(t)} S(T) + c'(\beta(t)).$$

$$C(\overbrace{\beta}^{\in L^2}) \in L^2[0, T] \\ = \int_0^T c(\beta(t)) dt. \quad \text{On écrit } \frac{\partial C}{\partial \beta}$$

$$\beta \in L^2 \rightarrow C(\beta) \in \mathbb{R}$$

$$\text{Si } C(\beta + \delta \beta) - C(\beta) = \left\langle \sum_{L^2}, \delta \beta \right\rangle + o(\|\delta \beta\|_{L^2})$$

$$\text{alors } \sum = \text{notre désiré. En fait } \sum = c'(\beta(0))$$

$$\leftarrow C'(\beta(t)) \quad \sum \in L^2.$$

$$\int_0^T C(\beta(t) + \delta\beta(t)) dt - \int_0^T C(\beta(t)) dt = \int_0^T c'(\beta(t)) \cdot \delta\beta(t) dt +$$

$\left. + o(||\delta\beta||_{L^2})\right).$

$$L(S, I, \lambda, \mu, \beta) = S(t) - \int_0^T \left(S(t) + \frac{\beta(t)S(t)I(t)}{N_p} \right) \lambda(t) dt$$

$$- \int_0^T \left(I^t - \frac{\beta SI}{N_p} + \delta I \right) \mu(t) dt$$

$$\frac{\partial}{\partial} \chi_1 \frac{\partial}{\partial} \mu : (SI) \checkmark$$

$$\frac{\partial}{\partial \beta(t)} = - \frac{S(t) \lambda(t)}{N_p} + \frac{S(t) I(t)}{N_p} \mu(t) = (\mu - \lambda) \frac{SI}{N}$$

$$L \stackrel{\text{IPP}}{=} S(t) - S \lambda \Big|_0^T + \int_0^T \left(S(t) \chi'(t) - \lambda \frac{\beta SI}{N_p} \right) dt$$

$$- I \mu \Big|_0^T + \int_0^T \left(I(t) \mu'(t) + \frac{\beta SI}{N_p} \mu - \delta I \mu \right) dt$$

Donc χ_1, μ vérifient les EDO :

$$\frac{\partial}{\partial t} \chi_1$$

$$\chi'(t) = \frac{\beta I}{N_p} (\lambda - \mu)$$

$$\frac{\partial}{\partial t} \mu$$

$$\boxed{\lambda(t) \approx 1}$$

$$\frac{\partial}{\partial t} \mu$$

$$\mu'(t) = \frac{\beta S}{N_p} (\lambda - \mu) + \delta \mu$$

$$\boxed{\mu(t) = 0}$$

$B_E = u \cdot a$.

B = processus sto

$$\{B_{t_1}, t \geq 0\} = (B_t)_{t \geq 0}$$

$$B_t - B_s \sim \sqrt{t-s} \cdot N(0, 1)$$

$$\sim N(0, t-s)$$

t	10^6	1	10^{-6}	10^{-6}
\sqrt{t}	10^3	1	10^{-2}	10^{-3}

$$\frac{1}{\sqrt{t}} \rightarrow T \quad N \cdot \left(\frac{T}{N} \right)^2 \rightarrow T$$

$$\int_0^t f(t) dt \quad \text{---} \quad \sum f(\xi_i) [t_{i+1} - t_i] \quad \xi_i \in [t_i, t_{i+1}]$$

$\xi_i = t_i$

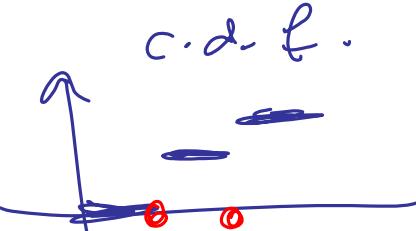
$$F(b) = F(a) + \int_a^b F'(x) dx$$

$$x=0, \mu=1 \quad x=B \quad B_x = B_0 + \int_0^x B'_x dx$$

$$F(b) = F(a) + \int_a^b dF(x)$$

measure ~~discrete~~ dont F ist la

$$[F(dx)]$$



$$\int f dg = \int f g' dx$$

$$x \in C^1 \quad f(x_t) = f'(x_t) x'_t .$$

$$f(g(t))' = f'(g_t) g'_t$$

$$df \circ g = f' \cdot dg$$

terme d'Intégration

$$d f(t, x_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx_t$$

$\left[\sum \frac{\partial f}{\partial x_i} H_x^i dt \right]$

$\underbrace{\frac{\partial f}{\partial x} dx_t}_{dt}$

$\boxed{\frac{\partial f}{\partial x} dx_t + \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} H_x dt}$

$$f(t, \tilde{x}) \approx x^2 \quad \frac{\partial f}{\partial t} = 0 \quad \frac{\partial f}{\partial x} = 2x . \quad + 1 \cdot d\tilde{x}$$

$$\tilde{x} = B$$

$$dB_t^2 = 0 \cdot dt + 2B_t dB_t + \frac{1}{2} \cdot 2 \cdot 1^2 dt$$

$$d(B_t)^2 = 2B_t dB_t + dt$$

$$(w_1) \quad \lim_{n \rightarrow \infty} \# \left\{ \frac{|y_{na} - y_n|}{h} \mid a \in A_{t_n} \right\} - a_n = 0$$

$$(w_2) \quad \lim_{n \rightarrow \infty} \# \left\{ \frac{(y_{na} - y_n)^2}{h} \mid a \in A_{t_n} \right\} - b_n^2 \stackrel{L^2}{\longrightarrow} 0.$$

$$\boxed{a_n, b_n \in A_{t_n}} \quad \hat{e}$$

$$\Delta w_r = w_{t_{r+1}} - w_{t_r} \sim d\int (0, t_{r+1}, t_r) = d(\theta h)$$

Pour montrer que $\lambda a_n \xrightarrow{n \rightarrow \infty} 0$ il faut montrer

$$\mathbb{E} \{ (\lambda a_n)^2 \} \xrightarrow{n \rightarrow \infty} 0$$

$$\Rightarrow w^2 \# \{ a_n^2 \} \left(\text{et } a_n \text{ est borné} \right) \xrightarrow{n \rightarrow \infty} 0$$

$$\mathbb{E} \left\{ \frac{\Delta w_n^{\epsilon-h}}{\sqrt{h}} \mid \mathcal{A}_{t_n} \right\} =$$

$$= \mathbb{E} \left\{ \Delta w_n^{\epsilon-h} \right\} \quad (\text{car } \Delta w_n \perp \mathcal{A}_{t_n})$$

$\underbrace{\mathbb{E} \left\{ \Delta w_n^{\epsilon-h} \right\}}$
o car $\Delta w_n \sim \mathcal{N}(0, h)$.

$$\mathbb{E} \left\{ \Delta w_n \frac{\Delta w_n^{\epsilon-h}}{\sqrt{h}} \right\} = \mathbb{E} \left\{ \frac{\Delta w_n^{\epsilon}}{\sqrt{h}} \right\} - \sqrt{h} \underbrace{\mathbb{E} \left\{ \Delta w_n \right\}}_{=0}$$

$$\Delta w_n = \sqrt{h} Z_n \quad Z_n \sim \mathcal{N}(0, 1),$$

$$\boxed{(\mathcal{F}_t) : \lim_{n \rightarrow \infty} \frac{Y_{n+h} - \mathbb{E} Y_{n+h} \mid \mathcal{A}_{t_n} - b_n \Delta w_n}{\sqrt{h}} \stackrel{\mathcal{L}}{\rightarrow} }$$

(Rg 4.18) la sol exacte est faiblement constante (satisfait w_t et w_0)

$$(w_t) : \mathbb{E} \left\{ \frac{X_{n+h} - X_n}{h} \mid \mathcal{A}_{t_n} \right\} - q(t_n, X_n)$$

$$\mathbb{E} \left\{ \frac{\int_{t_n}^{t_n+h} a(s, X_s) ds + \int_{t_n}^{t_n+h} b(s, X_s) dW_s}{h} \mid \mathcal{A}_{t_n} \right\} - q(t_n, X_n) = \dots$$