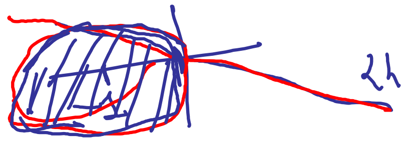
$U_{n+1} = U_n + h\underbrace{f_n}_{\lambda U_n} = (1+h\lambda)U_n = (1+h\lambda)(1+h\lambda)U_{n-1} =$

$= \dots = (1+h\lambda)^{n+1}U_0 \longrightarrow 0$ ssi $|1+h\lambda|<1$.

$\iff \text{dist}(h\lambda, -1) < 1$

En particulier si $|1+h\lambda|>1$ alors $|U_{n+1}| \to \infty$

ça arrive si h est trop grand !



$\{h\lambda \mid h \in \mathbb{R}\}$

Si $Re(\lambda) < 0$ $\underbrace{\text{sol exacte}}$ $\to 0$

$$e^{\lambda t} = e^{-|Re(\lambda)|t}\left[\cos(Im\,\lambda)t + i\sin(Im\,\lambda t)\right] \to 0$$

Si h trop grand divergence numérique.

Stabilité CN

$U_{n+1} = U_n + \frac{h}{2}[f_n + f_{n+1}] = U_n + \frac{h}{2}\lambda[U_n + U_{n+1}]$ donc

$U_{n+1}\left(1-\frac{h\lambda}{2}\right) = U_n\left(1+\frac{h\lambda}{2}\right)$ $\quad U_{n+1} = \dfrac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}} U_n = \dots$

$= \left(\dfrac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right)^{n+1} U_0 \longrightarrow 0$ ssi $\left|\dfrac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right| < 1$ i.e. $\underline{Re(\lambda) < 0}$

$\iff \left|1+\frac{h\lambda}{2}\right| < \left|1-\frac{h\lambda}{2}\right|$

Heun $U_{n+1} = U_n + \frac{h}{2}[f_n + f(t_n+h, U_n + hf_n)]$

$= U_n + \frac{h}{2}[\lambda U_n + \lambda(U_n + hf_n)] = U_n + \frac{h}{2}[\lambda U_n + \lambda U_n + h\lambda \overbrace{\lambda U_n}^{f_n}]$

$= U_n\left(1 + \frac{h\lambda}{2}\cdot 2 + \frac{(h\lambda)^2}{2}\right) \overset{z=h\lambda}{=} U_n\left(1+z+\frac{z^2}{2}\right)$

$= \dots = \left(1+z+\frac{z^2}{2}\right)^{n+1} \cdot U_0$.

Si $G \in C^2$ $o(\|h\|)$ ds Taylor devient $O(\|h\|^2)$

$h = (h_1, h_2)$

Prop 2.24 (FR) 2.32 (EN)

$$\tau_{n+1}(h) = \frac{X_{n+1} - U^{*}_{n+1}}{h} = \frac{X_{n+1} - [X_n + h \sum b_i K_i]}{h}$$

$K_i = f(t_n + c_i h, X_n + h \sum a_{ij} k_j)$

On doit montrer (cf Def 2.10 (EN) FR) que $\tau(h) = o(1)$, on va montrer $\tau(h) = O(h)$.

$K_i = f(t_n, X_n) + O(h)$

$$\tau_{n+1}(h) = \frac{X_{n+1} - X_n - h(\sum_{i=1}^{s} b_i) f(t_n, X_n) + h\,O(h)}{h}$$

$$= \frac{X_n + hX'_n + O(h^2) - X_n - h(\sum b_i) X'_n}{h} + O(h)$$

$= X'_n(1 - \sum_1^s b_i) + O(h)$ Il faut $\sum_1^s b_i = 1$

Methodes multi-pas : "économisent les calculs de f"

Preuve Thm 2.38 On veut montrer $\tau(h) \to 0$; on va montrer $\tau(h) = O(h)$.

$$\tau_{n+s}(h) = \frac{\sum_0^s a_k X(t_{n+k}) - h \sum_{k=0}^{s} b_k f(t_{n+k}, X_{n+k})}{h}$$

$$\frac{1}{h}\Big\{ \sum_{k=0}^{s} a_k [X(t_{n+s}) + X'(t_{n+s})\cdot(t_{n+k} - t_{n+s}) + O(h^2)]$$

$$- h \sum_{k=0}^{s} b_k [\underbrace{f(t_{n+s}, X_{n+s})}_{= X'(t_{n+s})} + O(h)] \Big\}$$

$$= \frac{1}{h}\Big\{ \Big(\sum_{k=0}^{s} a_k\Big) X(t_{n+s}) + h\Big[\sum_k a_k \cdot (k-s) - b_k\Big] X'(t_{n+s})$$

$$+ O(h^2) \Big\} = \frac{1}{h} X(t_{n+s}) \underbrace{\left(\sum_{k=0}^{s} a_k\right)} + \left(\sum_{k=0}^{s} a_k (k-s) - b_k\right) X'(t_{n+s})$$

$$+ O(h).$$

Pour la consistence il faut et il suffit que l'expression soit $O(h)$. En particulier ceci est vrai pour

$$\sum_{k=0}^{s} a_k = 0 \qquad \underbrace{\sum_{k=0}^{s} a_k (k-s) - b_k = 0}$$

$$\Leftrightarrow \sum_{k=0}^{s} a_k = 0 \quad , \quad \sum_{k=0}^{s} k a_k = \sum_{k=0}^{s} b_k + s \cdot \underbrace{\sum_{k=0}^{s} a_k}_{0}$$

$$\Leftrightarrow \sum_{k=0}^{s} a_k = 0 \quad , \quad \sum_{k=0}^{s} k a_k = \sum_{k=0}^{s} b_k .$$

SIR Il est possible de montrer $S(t) \geq 0$, $I(t) \geq 0$

$R(t) \geq 0$, $\qquad \frac{S(0)}{N} > \frac{1}{\mathcal{R}_0}$?

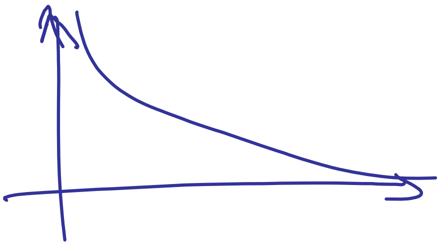
Proportion v à vacciner ? $\qquad$ Ex $\mathcal{R}_0 \simeq 15$

$S(0) = (1-v) N \qquad \frac{S(0)}{N} = 1 - v \leq \frac{1}{\mathcal{R}_0} \Leftrightarrow 1 - \frac{1}{\mathcal{R}_0} \leq v$

On veut

Il faut donc vacciner $1 - \frac{1}{\mathcal{R}_0}$. (Ex polio 95%)

$$\begin{cases} \dot S = -\beta S I \\ \dot I = \beta S I - \gamma I \end{cases}$$



$$\frac{\frac{dI}{dt}}{\frac{dS}{dt}} = \frac{\beta S I - \gamma I}{-\beta S I} = -1 + \frac{\gamma}{\beta}\,\frac{1}{S}$$

"$\frac{dI}{dS}$"

$$\boxed{I'(S) = -1 + \frac{1}{S}\cdot\frac{1}{R_0}}$$

$$I(u) = I(S_0) + \int_{S_0}^{u} -1 + \frac{1}{R_0}\,\frac{1}{S}\,dS$$

$$\boxed{I_t = I_0 + S_0 - S_t + \frac{1}{R_0}\ln\left(\frac{S_t}{S_0}\right)}$$

$t \to \infty \Rightarrow$ eq pour Z (taille de l'épidémie)

$$0 = I_0 + \underbrace{S_0 - S_\infty}_{Z} + \frac{1}{R_0}\ln\left(\frac{S_\infty}{S_0}\right)$$

$$S_\infty = S_0 - Z$$

$$I_0 + Z = \frac{1}{R_0}\ln\left(\frac{S_0}{S_0 - Z}\right) \Rightarrow R_0(I_0 + Z) =$$

$$\ln\left(\frac{S_0}{S_0 - Z}\right) \Rightarrow \exp\left(R_0(I_0 + Z)\right) = \frac{S_0}{S_0 - Z} \Rightarrow$$

$$1 - \frac{Z}{S_0} = \exp\left(-R_0(I_0 + Z)\right).$$